

lec 16 09 / 03 / 16 (Wed)

①

$$\begin{aligned}\partial_t^2 V_a(t) &= \partial_t \cdot 2 \operatorname{Im} \int \nabla u \cdot \nabla a \bar{u} dx \\ &= 4 \int \operatorname{Re}(\partial_\alpha u \partial_\beta \bar{u}) \partial_\alpha \partial_\beta a + 2\lambda \frac{p-1}{p+1} \int |u|^{p+1} \Delta a \\ &\quad - \int |u|^2 \Delta^2 a\end{aligned}$$

$a = |x| \Rightarrow \Delta^2 a \leq 0. \quad (\text{d} \geq 3)$

$$\Rightarrow \partial_t \cdot 2 \operatorname{Im} \int \nabla u \cdot \frac{x}{|x|} \bar{u} = \boxed{4 \int \frac{|\nabla u|^2}{|x|} dx} \geq 0$$

$+ 2\lambda \frac{p-1}{p+1} \int \frac{|u|^{p+1}}{|x|} dx$

$- \int |u|^2 \Delta^2 a$

$\underbrace{\quad}_{\geq 0.}$

$\Delta^2 a = -8\pi \delta \quad \text{when } d=3$

$\lambda = 1$ (defocusing)

$$\int_{t_0}^{t_1} \int \frac{|u|^{p+1}}{|x|} dx \lesssim \sup_{t=t_0, t_1} \left| \operatorname{Im} \int \nabla u(t) \cdot \frac{x}{|x|} \bar{u}(t) dx \right|$$

$$\lesssim \sup_{t_0, t_1} \|u(t)\|_{\dot{H}^{1/2}}^2$$

or $\leq \underset{\text{C-S}}{M(u_0)^{1/2} H(u_0)^{1/2}}$

Take $t_0 \rightarrow -\infty$, $t_1 \rightarrow +\infty \Rightarrow$ Morawetz estimate.

(2)

Pf of claim: $\partial_j \partial_k a \operatorname{Re}(\partial_j u \partial_k \bar{u}) = \frac{|\nabla u|^2}{|x|}$

$$\partial_j \partial_k a = \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3}$$

$$\begin{aligned} \Rightarrow (\text{LHS}) &= \sum_{j=1}^d \frac{|\partial_j u|^2}{|x|} - \sum_{j,k} \operatorname{Re} \left(\frac{x_j \partial_j u}{|x|}, \frac{x_k \partial_k \bar{u}}{|x|} \right) \frac{1}{x} \\ &= \frac{|\nabla u|^2}{|x|} - \frac{1}{|x|} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 = \frac{1}{|x|} \left| \nabla u - \frac{x}{|x|} \cdot \nabla u \right|^2 \\ &= \frac{|\nabla u|^2}{|x|} \quad (x - P_r x)^2 = |x - P_r x|^2 \end{aligned}$$

• Interaction Morawetz estimate (Colliander-Keel-Staffilani-Takaoka-Tao med'03)

Write ④ centered at y $d=3$

$$\begin{aligned} \partial_t \operatorname{Im} \int \nabla u(x) \cdot \frac{x-y}{|x-y|} \bar{u}(x) dx &= 2 \int \frac{|\nabla_y u(x)|^2}{|x-y|} dx + 2 \lambda \frac{p-1}{p+1} \int \frac{|u(x)|^{p+1}}{|x-y|} dx \\ &\quad + 4\pi |u(y)|^2 \end{aligned}$$

**

Multiply ~~∇u~~ by $|u(y)|^2$ and $\int_y dy$. ③

\Rightarrow

$$\int_{\mathbb{R}^t} \int_{\mathbb{R}^y} |u(y)|^4 dy dt \lesssim \sup_t \|u(t)\|_L^2 \|u(t)\|_{H^{1/2}}^2$$

or

$$M(u_0)^{3/2} H(u_0)$$

5.4 Scattering for (energy-subcrit) cubic NLS on \mathbb{R}^3 .

We only consider $t \rightarrow +\infty$.

WTS: $\exists u_+ \in H^1(\mathbb{R}^3)$ st. $\|u(t) - S(t)u_+\|_{H^1} \rightarrow 0$ as $t \rightarrow +\infty$.

$$\|S(t)u(t) - u_+\|_{H^1}$$

$$\underbrace{S(-t)u(t)}_{u_+} = u_0 - i \int_0^t S(-t')|u|^2 u(t') dt'$$

$\downarrow ?$

\Rightarrow Suffices to make sense (in H^1) of

$$\int_0^\infty S(-t)|u|^2 u(t) dt.$$

(4)

Existence of wave operator: $\Omega_+ : u_+ \in H' \rightarrow u_0 \in H'$.

Given u_+ in H' , can we find $u_0 \in H'$ s.t.

the corresp soln u scatters to $S(t)u_+$.

Rmk.. If Ω_+ exists, it is injective (by the uniqueness part of WP theory.)

- If Ω_+ is invertible, we say we have asymptotic completeness.

- $u_+ = u_0 - i \int_0^\infty S(-t)|u|^2 u(t) dt$

$$S(t)u(t) = u_0 - i \int_0^t S(-t')|u|^2 u(t') dt'$$

$$\oplus \Rightarrow u(t) = \underbrace{S(t)u_+}_{\text{value at } t=+\infty} + i \int_t^\infty S(t-t')|u|^2 u(t') dt' \quad \text{Terminal value problem.}$$

Pf of existence of wave op:

$$t = +\infty \rightarrow t = T \rightarrow t = 0$$

① well-posedness on $[T, \infty)$

$$\tilde{S}' = L_{t,x}^5 \cap L_t^{\frac{10}{3}} W_x^{1, \frac{10}{3}} \quad (\frac{10}{3}, \frac{10}{3}), \text{ adm.}$$

$$\cdot \|u\|_{L_{t,x}^5} \stackrel{\text{Sob}}{\lesssim} \|u\|_{L_t^5 W_x^{1, \frac{30}{11}}} \quad (5, \frac{30}{11}), \text{ adm.}$$

By Strichartz, $\|S(t) u_+ \|_{\tilde{S}'(R_t)} \lesssim \|u_+\|_{\dot{H}^1} < \infty$.

\Rightarrow By MCT

$$\|S(t) u_+ \|_{\tilde{S}'([T, \infty))} \leq \varepsilon.$$

Define $\tilde{\Gamma}(u(t))$ by $\tilde{\Gamma}(u(t)) = (\text{RHS}) \text{ of } \oplus$

$$\begin{aligned} \Rightarrow \|\tilde{\Gamma}(u)\|_{\tilde{S}'([T, \infty))} &\leq \varepsilon + C \underbrace{\|\nabla(|u|^2 u)\|}_{L_{t,x}^{10/7}([T, \infty))} \\ &\leq \|u\|_{L_{t,x}^5}^2 \|\nabla u\|_{L_{t,x}^{10/3}} \\ &\leq \|u\|_{\tilde{S}'([T, \infty))}. \end{aligned}$$

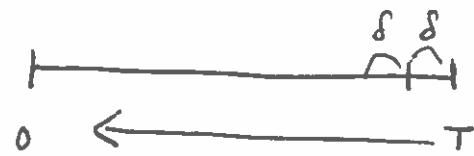
$$\text{Similarly, } \|\tilde{\mathcal{P}}u - \tilde{\mathcal{P}}v\|_{\tilde{S}'([T, \infty))} \leq C \left(\|u\|_{\tilde{S}'([T, \infty))}^2 + \|v\|_{\tilde{S}'([T, \infty))}^2 \right) \quad (6)$$

$\rightarrow \tilde{\mathcal{P}}$ is a contraction

$$\text{on } \{u : \|u\|_{\tilde{S}'([T, \infty))} \leq 2\varepsilon\}. \quad \varepsilon \ll 1.$$

Now, apply LWP and conservation of energy (& mass)

to extend u onto $[0, T]$.



$$\|u(t)\|_{H^1} \lesssim (H(u(\tau)) + M(u(\tau)))^{1/2}$$

$$\rightarrow \delta \sim (H(u(\tau)) + M(u(\tau)))^{-\theta}.$$



(7)

• Scattering (asymptotic completeness)

$$\begin{aligned}
 \left\| \int_0^{10} S(t) |u|^2 u(t) dt \right\|_{H^1} &\stackrel{\text{dual str}}{\lesssim} \| \langle \nabla \rangle (|u|^2 u) \|_{L_{t,x}^{10/3}} \\
 &\lesssim \|u\|_{L_{t,x}^5}^2 \|u\|_{L_t^{10/3} W_x^{1,10/3}} \\
 &\leq \left(\sup_{\substack{(q,r) \\ \text{adm}}} \| \langle \nabla \rangle u \|_{L_t^q L_x^r} \right)^3 =: \|u\|_{S^1}^3.
 \end{aligned}$$

So, it suffices to show $\|u\|_{S^1} \lesssim 1$. \rightarrow scattering.

Claim: "weak" space-time bd "strong" space-time bound

$$\|u\|_{L_{t,x}^q} \lesssim 1 \quad \text{for some } q \in \left[\frac{10}{3}, 10 \right]$$

implies "strong" space-time bd.: $\|u\|_{S^1} \lesssim 1$

Rmk: • Interaction Morawetz \Rightarrow "weak" space-time bd ($q=4$)
• Morawetz estimate in the radial setting

$$\Rightarrow \|u\|_{L_{t,x}^5} \lesssim 1$$

(8)

\Leftarrow Radial Sobolev Ineq: $\| |x|^s |u| \|_{L_x^\infty(\mathbb{R}^d)} \lesssim \| u \|_{H^1}$

$$\frac{d}{2}-1 \leq s \leq \frac{d-1}{2} \quad u, \text{radial.}$$

$\left(\Leftarrow \text{localize around fixed } x, \text{ apply 1-d Bragliardo-Nirenberg (in } r\text{).}$

polar coord

$$\int_+^\infty \int_x |u|^5 = \int_t^\infty \int_x |x| |u| \cdot \frac{|u|^4}{|x|} dx dt \leq \underbrace{\| |x| |u| \|_{L_{t,x}^\infty}}_{\text{rad Sob.}} \underbrace{\int_{t,x} \frac{|u|^4}{|x|}}_{\text{Morawetz}} \leq C(\| u_0 \|_{H^1})$$

Pf & Claim: Given $\varepsilon > 0$, divide $\mathbb{R}_+ = \bigcup_{j=1}^N I_j$ s.t. $\| u \|_{L_{I_j}^q L_x^q} \leq \varepsilon$.

Let $I_j = [t_j, t_{j+1})$.

$$\| u \|_{S^1(I_j)} \stackrel{\text{sr.}}{\lesssim} \| u(t_j) \|_{H^1} + \| u \|_{L_{t,x}^5}^2 \| u \|_{L_t^{\frac{10}{3}} W_x^{1, \frac{10}{13}}}^{\frac{10}{13}}$$

$$u(t) = S(t-t_j) u(t_j)$$

$$-i \int_{t_j}^t S(t-t') \| u \|^2 u(t) dt'$$

$$\leq \| u \|_{S^1(I_j)}$$

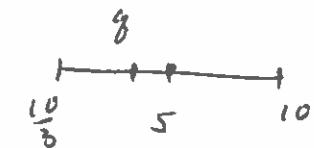
(A)

Note: . $\|u\|_{L_{t,x}^{10/3}(I)} \leq \|u\|_{S^0(I)}$ $(\frac{10}{3}, \frac{10}{3})$, adm

$$\leq \|u\|_{S^1(I)}$$

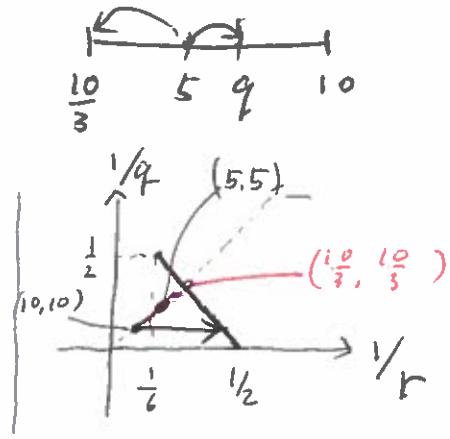
$\|u\|_{L_{t,x}^{10}(I)} \lesssim_{\text{Sob}} \|u\|_{L_I^{10} W_x^{1, \frac{30}{13}}} \leq \|u\|_{S^1(I)}$

\Rightarrow By interpolation, $\|u\|_{L_{t,x}^5(I_j)} \leq \|u\|_{L_{t,x}^8(I_j)}^\theta \|u\|_{L_{t,x}^{10}}^{1-\theta}$



or $\|u\|_{L_{t,x}^q}^\theta \|u\|_{L_{t,x}^{10/3}}^{1-\theta}$

B $\Rightarrow \|u\|_{L_{t,x}^5(I_j)} \leq \varepsilon^\theta \|u\|_{S^1(I_j)}^{1-\theta}$



$\Rightarrow \textcircled{A} \& \textcircled{B} \Rightarrow \|u\|_{S^1(I_j)} \lesssim \|u(t_j)\|_{H^1} + \varepsilon^{2\theta} \|u\|_{S^1(I_j)}^{3-2\theta}$

\Rightarrow continuity arg $\|u\|_{S^1(I_j)} \lesssim 1 \Rightarrow$ sum over finitely many intervals

$\|u\|_{S^1([0,\infty))} \lesssim 1.$

\Rightarrow scattering.

(10)

③

Actually, we can prove

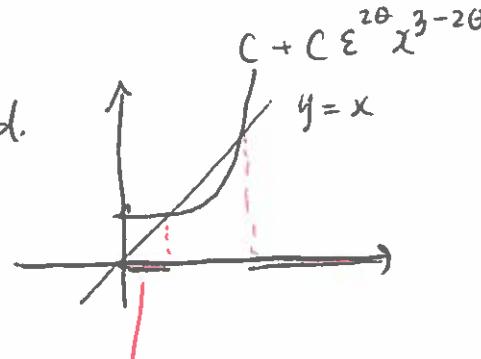
$$X(t) \lesssim 1 + \varepsilon^{2\theta} X(t)^{3-2\theta}$$

- $X(t)$ is conti.
- At $t = t_j$, is satisfied.

$$X(t_j) \lesssim 1$$

$$X(t) = \|u\|_{S^1([t_j, t])}$$

$$t \leq t_{j+1}$$



Here at time $t = t_j$,

$$\Rightarrow X(t) \lesssim 1 \quad (\text{if } \varepsilon \ll 1.)$$