

• Focusing mass-crit NLS

① If $\|u_0\|_{L^2} < \|Q\|_{L^2}$, (Q = ground state)

$M(u_0) < M(Q)$ then (NLS) is globally well-posed in $H^1(\mathbb{R}^d)$.

Moreover, all solns scatter as $t \rightarrow \pm \infty$.

$M(u_0) = M(Q)$ ② $U(t) = e^{it} Q$ is a global non-scattering soln.

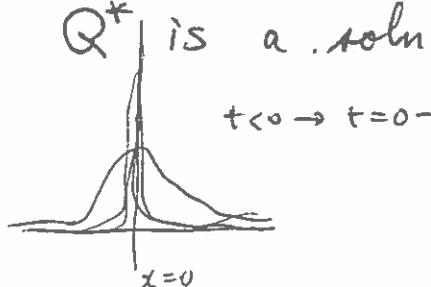
③ pseudo-conformal symmetry ($s_{\text{crit}} = 0$)

$$u(t, x) \longmapsto v(t, x) = \frac{1}{|t|^{d/2}} u\left(-\frac{1}{t}, \frac{x}{t}\right) e^{\frac{i|x|^2}{4t} (t+0)}$$

Apply pc transf. to $e^{it} Q$.

$$\Rightarrow Q^*(t, x) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{-\frac{i|x|^2}{4t} + \frac{i}{t}}$$

Q^* is a soln to (NLS) for $t < 0$ (and $t > 0$)



Q^* blows up at time $t=0$ (starting at $t=-1$) ②

• $|Q^*(t)|^2 \rightarrow \|Q\|_{L^2}^2 \delta_{x=0}$ as $t \nearrow 0$.

• $\|\nabla Q^*(t)\|_{L^2} \sim \frac{1}{|t|}$ \Leftarrow blowup speed

Q^* not "stable"

$\Leftarrow Q^*$ is the minimal mass blowup soln.

• unique (Merle '93)

i.e. if $M(u) = M(Q)$ and blows up in a finite time,

then $u = Q^*$ (up to symmetry)

• Other finite time blowup solns?

$$M(Q) < M(u_0) < M(Q) + \varepsilon.$$

Merle - Raphaël '00's ~

"log log" blowup soln $\sim \sqrt{\frac{\log \log(T-t)}{T-t}}$
 \Leftarrow "stable"

Rmk: slowest possible blowup speed $\gtrsim \frac{1}{\sqrt{T-t}}$ (by scaling)

5.3

Virial identity & Morawetz estimate

Viriel = "force"

$$i\partial_t u + \Delta u = \lambda |u|^{p-1} u \quad \lambda = \pm 1$$

$$(\partial_t u = i\Delta u - i\lambda |u|^{p-1} u)$$

Virial potential $V_\alpha(t) = \int a(x) |u|^2 dx$, real $\alpha(x)$ "nice" func

$$\cdot \alpha(x) = |x|^2$$

$$\alpha(x) = |x|$$

$$\begin{aligned} \partial_t V_\alpha(t) &= \int a(x) 2 \operatorname{Re}(\partial_t u \bar{u}) dx \\ &= -2 \operatorname{Im} \int a \Delta u \bar{u} dx - 2\lambda \int a(x) \operatorname{Re}(i|u|^{p+1}) dx \\ &\stackrel{\text{IBP}}{=} 2 \operatorname{Im} \int \nabla u \cdot \nabla(a \bar{u}) dx \\ &= 2 \operatorname{Im} \int \bar{u} \nabla u \cdot \nabla a dx \end{aligned}$$

$$\operatorname{Im} \int (\nabla u \cdot \nabla \bar{u}) a dx = 0$$

$$|\nabla u|^2$$

Morawetz action.

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• Compute $\partial_t^2 V_a(t)$

$$\partial_t 2 \int \bar{u} \nabla u \cdot \nabla a \, dx = 2 \int \partial_t \bar{u} \nabla u \cdot \nabla a + 2 \int \bar{u} \nabla \partial_t u \cdot \nabla a$$

$$= 2 \int (-i\Delta \bar{u}) \nabla u \cdot \nabla a + 2i \int \lambda |u|^{p-1} \bar{u} \nabla u \cdot \nabla a$$

$$+ 2 \int \bar{u} \nabla (i\Delta u) \cdot \nabla a - 2i \int \bar{u} \nabla (\lambda |u|^{p-1} u) \cdot \nabla a$$

$$|u|^{p-1} = u^{\frac{p-1}{2}} \bar{u}^{\frac{p-1}{2}}$$

$$= 2i\lambda \int |u|^{p-1} \bar{u} \partial_j u \partial_j a$$

Einstein's summation notation

$$- 2i\lambda \left\{ \int \frac{p+1}{2} |u|^{p-1} \partial_j u \bar{u} \partial_j a + \frac{p-1}{2} |u|^{p-1} u \partial_j \bar{u} \partial_j a \right\}$$

$$\partial_i u \partial_j a = \sum_{j=1}^d \partial_j u \partial_j a$$

$$= -2i\lambda \frac{p-1}{2} \int |u|^{p-1} (\partial_j u \bar{u} + \partial_j \bar{u} u) \partial_j a$$

$$= -2i\lambda \frac{p-1}{p+1} \int 2\bar{u} (|u|^{p+1}) \partial_j a \stackrel{\text{IBP}}{=} 2i\lambda \frac{p-1}{p+1} \int |u|^{p+1} \Delta a$$

$$(\bar{u} \bar{u})^{\frac{p+1}{2}}$$

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IBP on 2nd term.

$$\begin{aligned}
 &= -2i \int \Delta \bar{u} \nabla u \cdot \nabla a - 2i \int \Delta u \nabla \bar{u} \cdot \nabla a - 2i \int \Delta u \bar{u} da \\
 &= \textcircled{1} + \textcircled{2} + \textcircled{3}
 \end{aligned}$$

$$\textcircled{1} \stackrel{\text{IBP}}{=} 2i \int \partial_h \bar{u} \partial_h (\nabla u \cdot \nabla a) = 2i \cancel{\int \partial_h \bar{u} (\partial_h \nabla u) \cdot \nabla a} + 2i \int \partial_h \bar{u} \nabla u \cdot \partial_h \nabla a.$$

$$\textcircled{2} = 2i \int \partial_h u \partial_h (\nabla \bar{u} \cdot \nabla a) = \underbrace{2i \int \partial_h u (\partial_h \nabla \bar{u}) \cdot \nabla a}_{\substack{\text{IBP} \\ \equiv -2i \int \partial_h \bar{u} \partial_h \nabla u \cdot \nabla a}} + 2i \int \partial_h u \nabla \bar{u} \cdot \partial_h \nabla a$$

$$- 2i \cancel{\int \partial_h \bar{u} \partial_h u \Delta a}$$

$$\textcircled{3} = 2i \int \partial_h u \partial_h (\bar{u} \Delta a) = \cancel{2i \int \partial_h u \partial_h \bar{u} \Delta a} + 2i \int \partial_h u \bar{u} \partial_h \Delta a$$

Side comp: $2 \operatorname{Im} i \int \partial_h u \bar{u} \partial_h \Delta a = \frac{2i \int (\partial_h u) \bar{u} \partial_h \Delta a + 2i \int u (\partial_h \bar{u}) \partial_h \Delta a}{2i}$

$$= \int \partial_h (|u|^2) \partial_h \Delta a \stackrel{\text{IBP}}{=} - \int |u|^2 \Delta^2 a$$

$$\Rightarrow \partial_t^2 V_{\text{aff}} = \underbrace{4 \int \operatorname{Re}(\partial_h u \partial_j \bar{u}) \partial_h \partial_j a}_{-\int |u|^2 \Delta^2 a} + 2 \lambda \frac{p-1}{p+1} \int |u|^{p+1} \Delta a$$

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Ex: Virial Identity . $a(x) = |x|^2 = \sum_{j=1}^d x_j^2$. $\Delta a = 2d$
 $\Delta^2 a = 0$.

$$\Rightarrow V(t) = \int |x|^2 |u(t, x)|^2 dx \quad \partial_k \partial_j a = 2 \delta_{jk}$$

$$\text{and } \partial_t^2 V(t) = 8 \int |\nabla u|^2 + 4d \lambda \frac{p-1}{p+1} \int |u|^{p+1}$$

$$= 16 H(u) + \underbrace{\frac{4\lambda d}{p+1} \left(p - \left(1 + \frac{4}{d} \right) \right)}_{\text{mass-crit power.}} \int |u|^{p+1}$$

• focusing ($\lambda = -1$), $\text{Scrit} \geq 0$

$$\Rightarrow \boxed{\text{2nd term}} \leq 0$$

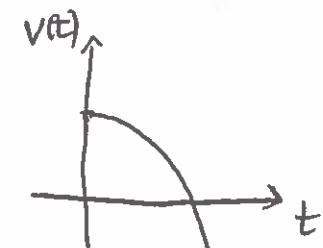
$$\text{Suppose } u_0 \in H^1(\mathbb{R}^d) \text{ s.t. } H(u_0) < 0. \Rightarrow \partial_t^2 V(t) \leq 16 H(u(t)) \\ = 16 H(u_0) < 0$$

$$\Rightarrow V(t^*) < 0 \text{ for some } t^* > 0.$$

$$\text{but } V(t) = \int |x|^2 |u(t, x)|^2 dx \geq 0$$

$\Rightarrow u$ must blow up before time t^* .

(Glassey's argument , Zakharov's)



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$$\cdot \underline{\text{Morawetz estimate}}: a = |x|, \partial_j a = \frac{x_j}{|x|}$$

$$\Rightarrow \partial_j^2 a = \frac{1}{|x|} - \frac{x_j}{|x|^2} \cdot \frac{\partial_j}{|x|} \Rightarrow \Delta a = \frac{d-1}{|x|}$$

$$\Rightarrow \partial_j(\Delta a) = (d-1) \frac{-x_j}{|x|^3} \Rightarrow \partial_j^2(\Delta a) = (d-1) \left(-\frac{1}{|x|^3} + \frac{3x_j}{|x|^4} \frac{x_j}{|x|} \right)$$

$$\Rightarrow \Delta^2 a = -\frac{(d-1)(d-3)}{|x|^3} \leq 0 \quad \text{if } d \geq 3.$$

$$\underline{d=3}: \frac{1}{|x|} = \text{fund. soln of } -\Delta \Rightarrow \Delta^2 a = -8\pi f.$$

Claim: $\partial_j \partial_k a \operatorname{Re} \partial_j u \partial_k \bar{u} = \frac{|\nabla u|^2}{|x|}$

angular component of grad

$$\nabla u = \nabla u - \frac{x}{|x|} \left(\frac{x}{|x|} \cdot \nabla u \right)$$

$$|\nabla u|^2 = |\nabla u|^2 - \left| \frac{x}{|x|} \cdot \nabla u \right|^2$$

Goal:

Morawetz: $\int_{\mathbb{R}^+} \int_{\mathbb{R}_x^d} \frac{|u(t, x)|^{p+1}}{|x|} dx dt \lesssim \sup_t \|u(t)\|_{H^{1/2}}^2$

$$\text{or } M(u_0)^{1/2} H(u_0)^{1/2}$$

defocusing case
 $\lambda = 1$