

- cubic NLS on \mathbb{R} , $s_{crit} = -1/2$ mass-subcritical
 GWP in $L^2(\mathbb{R})$ by iterating LWP & conservation of mass
 \Rightarrow holds for defocusing & focusing.

Rmk: We proved LWP of cubic NLS on \mathbb{R}^2 in $L^2(\mathbb{R}^2)$
 $s_{crit} = 0$. $\xrightarrow{\text{mass-critical}}$ critical space

local existence time $T \sim \|S(t)u_0\|_{L_T^4 L_x^4} (\ll 1)$

\uparrow can not be quantified in terms of $\|u_0\|_{L_x^2}$

\Rightarrow mass conservation alone does not yield GWP. (Dodson '12?)

- cubic NLS on \mathbb{R}^3 , $s_{crit} = 1/2$. energy-subcritical.

defocusing case: $H(u) + M(u) = \frac{1}{2} \int |\nabla u|^2 + \underbrace{\frac{1}{4} \int |u|^4}_{\geq 0} + \int |u|^2 \gtrsim \|u\|_{H^1}^2$

\Rightarrow GWP in $H^1(\mathbb{R}^3)$ $\left\{ \begin{array}{l} \text{local existence time} \\ T \sim \|u_0\|_{H^1}^{-\theta} \end{array} \right.$

• Let $u_0 \in H^1(\mathbb{R}^3)$.

(2)

$$\begin{aligned} \Rightarrow \|u(t)\|_{H^1} &\leq C \left(H(u(t)) + M(u(t)) \right)^{1/2} \\ &\stackrel{\text{cons}}{=} C \left(H(u_0) + M(u_0) \right)^{1/2} =: K < \infty, \quad \forall t \in \mathbb{R} \end{aligned}$$

use Sobolev

• Choose local existence time $T \sim K^{-\theta}$.

\Rightarrow Iterate LWP over intervals of length $T \rightarrow$ GWP.

but in the focusing case, $H(u) + M(u) \not\approx \|u\|_{H^1}^2$
and \exists finite time blowup solns.

• quintic NLS on \mathbb{R} : $S_{\text{crit}} = 0$. (mass-crit but energy-subcrit.)

mass conservation alone does not yield GWP in $L^2(\mathbb{R})$ (Dodson '15?)

defocusing: conservation of mass & energy \Rightarrow GWP in $H^1(\mathbb{R})$.

• energy-supercritical NLS, $S_{crit} > 1$

defocusing case: \mathbb{EWP} is open.

(analogous to NSE on \mathbb{R}^3 : $H^{\frac{1}{2}}$ -crit but energy $\int |u|^2 \approx L^2$ -norm)

(NLS) $i\partial_t u + \Delta u = -|u|^{p-1}u$ focusing

• Solitons (solitary wave solution)

$u(t, x) = e^{it} \underbrace{\phi(x)}_{\text{profile}}$

\Leftrightarrow solves (NLS) iff ϕ solves the following elliptic PDE:

(*) $\Delta \phi - \phi + |\phi|^{p-1}\phi = 0$, $\phi \in H^1(\mathbb{R}^d)$

FACT: $d=1$: all solns to (*) are translates of

$Q(x) = \left(\frac{p+1}{2 \cosh^2 \left(\frac{p-1}{2} x \right)} \right)^{p-1}$



• $d \geq 2$: \exists seq $\{Q_n\}_{n \geq 0}$ of real solns to $(*)$ with increasing L^2 -norms s.t.

(4)

$Q_n(r)$ vanishes n times on \mathbb{R}_+ .

Q_0 , radially sym, positive. \leftarrow ground state

uniqueness: $\phi > 0$, $\phi \in H^1$

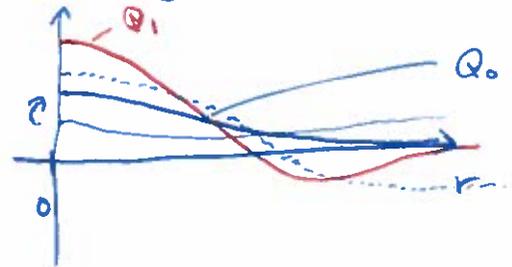
radial sym, C^2 , exp decaying

existence: Berestycki-Lions-Peletier '81.

Gidas-Ni-Nirenberg '79
Kwong '89

• Ground states play an important role in elliptic PDEs, dispersive PDEs, variational problems, functional inequalities, etc.

shooting method (on ODE, $r > 0$)



• mass - subcritical: $p < 1 + \frac{4}{d}$
scrit < 0

NLS scaling $Q^\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ ← ground state

Prop (variational characterization of Q)

$d \geq 1$, $1 < p < 1 + \frac{4}{d}$
 $M > 0$ fixed.

Then, the following minimization problem

$$\min_{\|u\|_{L^2} = M} H(u)$$

has min attained at $Q^{\lambda(M)}(\cdot - x_0) e^{i\sigma_0}$, $x_0 \in \mathbb{R}^d$, $\sigma_0 \in \mathbb{R}$
↗ rescale of Q s.t. $\|Q^{\lambda(M)}\|_{L^2} = M$ for all

(Lagrange multipl problem. $\frac{d}{d\varepsilon} H(u + \varepsilon v)|_{\varepsilon=0}$ ← Gateaux deriv.
⇒ Euler-Lagrange equation: $\Delta \phi - \lambda \phi + |\phi|^{p-1} u = 0$
↑ Lag. multipl

⑥

• mass-critical case = $S_{crit} = 0$. $p = 1 + \frac{4}{d}$

$$\text{let } J(u) = \frac{\left(\int |\nabla u|^2\right) \left(\int |u|^2\right)^{2/d}}{\int |u|^{2+4/d}}, \quad u \neq 0.$$

Prop (i) $\min_{\substack{u \in H^1 \\ u \neq 0}} J(u)$ is attained at

$$\lambda_0^{d/2} Q(\lambda_0 x + x_0) e^{i\tau_0}, \quad (\lambda_0, x_0, \tau_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$$

↑
unique ground state.

In particular, we have the sharp Gagliardo-Nirenberg inequality:

$$\int |u|^{2+4/d} \leq \underbrace{J(Q)}_{\substack{\uparrow \\ \text{optimal const.}}} \int |\nabla u|^2 \left(\int |u|^2\right)^{2/d}.$$

$$H(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{\frac{2+4}{d}} \int |u|^{\frac{2+4}{d}}$$

Rmk: $\forall u \in H^1$

⊗⊗

$$H(u) \geq \frac{1}{2} \int |\nabla u|^2 \left(1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^{4/d}\right) dx$$

"Rigidity".

(ii) Let $u \in H^1$ s.t.

$$H(Q) = 0.$$

(7)

$$\int |u|^2 = \int Q^2, \quad H(u) = 0.$$

Then,

$$u(x) = \lambda_0^{d/2} Q(\lambda_0 x - x_0) e^{i\delta_0}$$

for some $(\lambda_0, x_0, \delta_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$.

(For mass-subcrit,
 $H(Q) < 0$)

Cont'd Rmk:

mass-critical NLS,

1-d quintic
2-d cubic, etc.

focusing

Let $u_0 \in H^1$ s.t. $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

$$(*) \Rightarrow H(u) + M(u) \gtrsim \|u\|_{H^1}^2.$$

implicit const depends on $\|u_0\|_{L^2}$

\Rightarrow GWP in $H^1(\mathbb{R}^d)$ (also soln scatters)

provided that $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$

Note: mass-subcritical NLS is GWP in $L^2(\mathbb{R}^d)$ by LWP & mass conservation.