

lec 13 29/02/16 (Mon)

No lectures in the weeks
of Mar. 14, 21.

(P)

Sec 5: Global-in-time behavior of solns to NLS

We "proved" LWP of NLS if $s \geq \max(\text{crit}, 0)$.

Q1: Does the soln exist globally (in time)? Global well-posedness

Or does it cease to exist at (before) some finite time?

finite time blowup solutions

Q2: If u exists globally in time,
then what is the behavior of u as $t \rightarrow \pm\infty$?

- scattering: "asymptotic linear behavior." ($\Rightarrow \|u(t)\|_{L_x^\infty} \rightarrow 0$)

$\exists u_\pm \in H^s(\mathbb{R}^d)$ s.t. $\lim_{t \rightarrow \pm\infty} \|u(t) - \underbrace{s(t)u_\pm}_{H^s}\| = 0$.

- non scattering soln

such as soliton: $u(t) = e^{it} Q(x)$ (but \neq s(t)u_0)

\uparrow basically keeps the same profile.

(2)

Conjecture: Soliton resolution conjecture.

" $u(t)$ decouples into a sum of solitons.

+ radiation (= scattering part) as $t \rightarrow \pm\infty$.

Still open: except for "integrable equations" such as KdV.
and NLW (Kenig-Merle et al. '12?)

(5.1) Conservation laws: $i \partial_t u + \Delta u = \pm |u|^{p-1} u$

$$(\partial_t u = i \Delta u \mp i |u|^{p-1} u.)$$

mass: $M(u) = \int_{\mathbb{R}^d} |u|^2 dx$ \leftarrow for KdV, some may call it "energy".

Claim: $M(u(t)) = M(u_0)$ if u is a $\underbrace{\text{soln}}_{(\text{smooth})}$ to (NLS)

(For "rough" solns, we need to use well-posedness theory, conti dep.)

$$\begin{aligned} \partial_t \int_{\mathbb{R}^d} |u|^2 &= 2 \operatorname{Re} \int \partial_t u \cdot \bar{u} = 2 \operatorname{Re} i \int \Delta u \cdot \bar{u} \mp 2 \operatorname{Re} i \underbrace{\int |u|^{p+1}}_{\text{purely imaginary}} \\ &= -2 \operatorname{Re} i \int |\nabla u|^2 \\ &= 0. \end{aligned}$$

Rmk: mass $M \rightsquigarrow u \mapsto e^{i\theta} u$ (gauge invariance)

See HW3.

• Hamiltonian (energy): $H(u) = \underbrace{\frac{1}{2} \int |\nabla u|^2}_{\text{kinetic energy}} \pm \underbrace{\frac{1}{p+1} \int |u|^{p+1}}_{\text{potential energy}}$.

NLS can be written as
a Hamil dynamics: $\partial_t u = -i \frac{\partial H}{\partial \bar{u}}$

- + sign: defocusing. "lin dispersion & nonlin 'cooperate' \Rightarrow 'expect' GWP"

$$\Rightarrow H(u) \gtrsim \|u\|_{H^1}^2.$$

- sign: focusing $\Rightarrow H(u)$ may not control $\|u\|_{H^1}^2$.
↳ lin disp vs concentration by nonlinearity.

Critical regularity: $\text{Scrit} = \frac{d}{2} - \frac{2}{p-1}$.

- We say NLS is

- mass-critical if $\text{Scrit} = 0$. $p = 1 + \frac{4}{d}$ $1-d$: quintic
 $2-d$: cubic
 - energy-critical if $\text{Scrit} = 1$. $p = 1 + \frac{4}{d-2}$ $d=1, 2$: no energy crit.
 $3-d$: quintic
 $4-d$: cubic
- defocusing
- energy-subcritical if $\text{Scrit} < 1$
- energy-supercritical if $\text{Scrit} > 1$. (GWP of NLS/NLW is entirely open)

classical
 $H(p, q)$
 $\Rightarrow \partial_t p = \frac{\partial H}{\partial q}$
 $\partial_t q = -\frac{\partial H}{\partial p}$
 \downarrow
 $\begin{aligned} \partial_t \begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial H}{\partial (p, q)} \\ &\sim i \end{aligned}$

(4)

Claim: $H(u(t)) = H(u_0)$ if u is a soln to (NLS)

← related to time translation

$$u(t) \hookrightarrow u(t - t_0)$$

$$\begin{aligned} \partial_t \frac{1}{2} \int |\nabla u|^2 &= \operatorname{Re} \int \partial_t \nabla u \cdot \nabla \bar{u} \stackrel{\text{IBP}}{=} -\operatorname{Re} \int \partial_t u \Delta \bar{u} \\ &= -\operatorname{Re} i \int |\Delta u|^2 \cancel{+} \operatorname{Re} i \int |u|^{p-1} u \Delta \bar{u}. \end{aligned}$$

$$\begin{aligned} \pm \partial_t \left(\frac{1}{p+1} \int |u|^{p+1} \right) &= \pm \frac{1}{2} \int |u|^{p-1} \underbrace{\partial_t(|u|^2)}_{2 \operatorname{Re} \partial_t u \bar{u}} \\ &= \pm \operatorname{Re} i \int |u|^{p-1} \bar{u} \Delta u \cancel{-} \operatorname{Re} i \int |u|^{2p} \cancel{+} \operatorname{Re} i \int |u|^{p-1} u \Delta \bar{u} \end{aligned}$$

→ add to 0.

Aside: If $u \in C(\mathbb{R}; H^\infty)$, then u is also ∞ -differentiable in time.

$$\Leftarrow \text{use } \partial_t u = \underbrace{i \Delta u}_{C \cap H^\infty} \mp i \underbrace{|u|^{p-1} u}_{C \cap H^\infty} \text{ and repeat.}$$

by Sobolev

$$H^\infty = \bigcap_{s \in \mathbb{R}} H^s$$

Momentum

$$P(u) = \operatorname{Im} \int \bar{u} \nabla u \in \mathbb{R}^d.$$

$$= \int \frac{\bar{u} \nabla u - u \nabla \bar{u}}{2i} = -i \int \bar{u} \nabla u$$

~ related to spatial translation
 $u(t, x) \mapsto u(t, x + x_0)$

Claim: $P(u(t)) = P(u_0)$ if u is a soln to (NLS).

Aside: $\|u\|_{L_x^2} = 1$. $\Rightarrow |u(x)|^2 dx$ is a probability density.

• $\int x |u(x)|^2 dx$ = "expected position".

try to write it
on the Fourier side

$$\hookrightarrow \text{Galilean invariance}$$

$$u(t, x) \mapsto e^{i\beta \cdot x} e^{i|\beta|^2 t} u(t, x - \beta t)$$

• Plancked: $\|\hat{u}\|_{L_z^2} = 1$.

$\Rightarrow |\hat{u}(\xi)|^2 d\xi$ is also a prob. density.

shift by β
on the Fourier side

$$\beta \in \mathbb{R}^d$$

$$P(u) = \underset{\text{Parseval}}{\int \overline{\hat{u}(\xi)} \xi \hat{u}(\xi) d\xi} = \int \xi |\hat{u}(\xi)|^2 d\xi = \text{"expected velocity"} \\ \text{momentum}$$

(5.2) GWP and finite time blowup solns

- cubic NLS on \mathbb{R} . $\sigma_{\text{crit}} = -\frac{1}{2}$, i.e. mass-subcritical.

We proved LWP in $L^2(\mathbb{R})$, where local existence time $T \sim \|u_0\|_{L^2}^{-\theta}$, $\theta > 0$.
 (in the subcritical sense)

But conservation of mass says

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad (\text{while the LWP argument gives } \|u(T)\|_{L^2} \leq 2\|u_0\|_{L^2})$$

\Rightarrow can iterate LWP to construct global soln.

over time interval of length T .

