- defocusing quintic NLS

$$\begin{cases} i u_t + u_{xx} - u|u|^4 = 0 & , (x,t) \in \mathbb{T} \times \mathbb{R} \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}) \end{cases}$$

- LWP: Bourgain '93, $s > 0$

L^6- Strichartz estimate

$$\| S(t) u_0 \|_{L^6_{x,t}(\mathbb{T}^2)} \lesssim \| u_0 \|_{H^\varepsilon(\mathbb{T})}$$

$\left(S(t) = \text{lin. semigroup} : \widehat{S(t) u_0}(n) = e^{-itn^2} \widehat{u_0}(n) \right)$

$\Leftarrow$ fails for $\varepsilon = 0$.

- GWP: given subcritical LWP (i.e. $T \sim \| u_0 \|_{H^s}^{-\alpha}$), a priori bd on $\| u(t) \|_{H^s}$ yields GWP.

Hamiltonian

$$H(u) = \frac{1}{2} \int |u_x|^2 + \frac{1}{6} \int |u|^6$$

controls H^1-norm $\Rightarrow$ GWP in H^1.

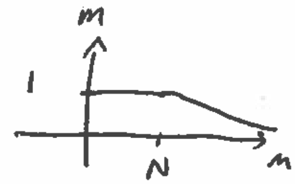
$$\left(\begin{array}{l} \| u \|_6^6 \gtrsim \| u \|_2^6 \gtrsim \| u \|_2^2 \quad \text{if } \| u \|_2 \gtrsim 1 \\ \lambda\!\!\!\wedge \\ \| u \|_{H^{1/3}}^6 \lesssim \| u \|_{L^2}^4 \| u \|_{H^1}^2 \lesssim \| u \|_{H^1}^2 \quad \text{if } \| u \|_2 \lesssim 1 \end{array} \right.$$

- **I-method**: $\quad iq_t = \dfrac{\partial H}{\partial \bar{q}}, \qquad q = \{q_n\}_{n\in\mathbb{Z}}$

Define $I : H^s \to H^1, \quad s<1$

by $\quad Iq_n = m(n)\, q_n,$

$$m(n) = \begin{cases} 1, & |n| \le N \\ \dfrac{N^{1-s}}{|n|^{1-s}}, & |n| \ge N \end{cases}$$



$$\Rightarrow \quad \|q\|_{H^s} \le \|Iq\|_{H^1} \le N^{1-s}\|q\|_{H^s}.$$

- Given $q(0) \in H^s$, we have

$$H(Iq(0)) \sim \|Iq(0)\|_{H^1}^2 \lesssim N^{2(1-s)}$$

- Iq solves

$$\begin{cases} i\, Iq_t + Iq_{xx} - I(q|q|^4) = 0 \\ Iq\big|_{t=0} = Iq(0) \in H^1 \end{cases}$$

- **Goal of I-method**:

Obtain a good estimate on $\left|\dfrac{d}{dt} H(Iq(t))\right|$

$$\Rightarrow \quad T\left|\frac{d}{dt} H(Iq(t))\right| \lesssim N^{2(1-s)}$$ (with abs. value ...)

T ← doubling time

$$\Rightarrow \quad H(Iq(t)) \lesssim N^{2(1-s)}, \qquad |t| \le T$$

- can iterate the local argument with a <u>fixed</u> time step size as long as ③

$$\| I q(t) \|_{H^1} \lesssim N^{1-s}$$

or $$H(I q(t)) \lesssim N^{2(1-s)}$$

(<u>Note:</u> GWP, $s > 4/9$ in De Silva - Pavlović - Tzirakis - Staffilani 07 is NOT correct.

<u>Idea:</u> Apply normal form reduction to H and then use the I-method.

$$H(q) = \underset{\underset{\sum n^2 |q_n|^2}{\parallel}}{H_0(q)} + N(q)$$

$$N(q) = \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n})\, q_{n_1} \bar{q}_{n_2} \cdots \bar{q}_{n_{2r}}$$

Let $D(\bar{n}) = | n_1^2 - n_2^2 + \cdots - n_{2r}^2 |$

Write $N(q) = N_0(q) + N_1(q)$

N_0: $D < K$ resonant

N_1: $D > K$ nonresonant.

Γ = Lie transform w.r.t. F

i.e. time 1 flow map of

$$i q_t = \frac{\partial F}{\partial \bar{q}}$$

$\Rightarrow$ By Taylor expansions (see Bambusi's note,)

$$H' = H \circ \Gamma^{-1} = H_0 + N_0 + \cancel{N_1}$$
$$\left.\begin{aligned} &+ \cancel{\{H_0, F\}} + \{N_0, F\} + \{N_1, F\} \\ &+ \frac{1}{2}\{\{H_0, F\}, F\} + \cdots \end{aligned}\right\} \text{h.o.t.}$$

$\Rightarrow$ Choose $\quad \{H_0, F\} = -N_1$

$$\Rightarrow \quad F = -\sum \frac{C(\bar{n})}{D(\bar{n})} q_{n_1} \bar{q}_{n_2} \cdots \bar{q}_{n_{2r}}$$

- Repeat the process to eliminate h.o.t.

Note: $\quad \|\Gamma q\|_{L^2} = \|q\|_{L^2}$

$$\|\Gamma q\|_{H^1} \sim \|q\|_{H^1}.$$

- Let $G(q) = \|q\|_2^2 = \sum |q_n|^2$

$$\Rightarrow \frac{\partial G}{\partial t} = c\{F, G\} = c(F - F) = 0$$

- $$\Gamma q = q(1) = q(0) + c\int_0^1 \frac{\partial F}{\partial \bar{q}}\, dt$$

<u>Note</u>: "t" denotes the time w.r.t. $i q_t = \frac{\partial F}{\partial \bar{q}}$, and has nothing to do with the time of NLS.

$$\Rightarrow \|q - \Gamma q\|_{H^1} \leq \left\| \frac{\partial F}{\partial \bar{q}} \right\|_{H^1}$$

duality variable

$$\lesssim \|q\|_{H^1} \sum \frac{c(\bar{n})}{D(\bar{n})} \underbrace{\boxed{\frac{|n_1| q_{n_1}}{\|q\|_{H^1}}} \bar{q}_{n_2} \cdot \cdots \cdot q_{n_{2r-1}} \cdot \boxed{p_{n_{2r}}}}_{\text{only in } L^2}$$

$$\lesssim \|F\| \, \|q\|_{H^1}$$

$$\underset{(3.14)}{\lesssim} N^{-\varepsilon} \|q\|_{H^1}$$

<u>Regularity</u>: $\Gamma = \Gamma_F$ acts boundedly on ℓ^2.
(See Kuksin-Pöschel '96 for $s > 1/2$.)

Define

$$\|u\|_{0,p} = \left[\sum_m \left(\sum_n |\hat{u}(n, n^2 + m)|^2\right)^{p/2}\right]^{1/p}$$

$$= \| S(-t) u \|_{\mathcal{F}L^p_t, L^2_x}$$

L^6- Strichartz estimate

$\Rightarrow$

(7.10) $$\|u\|_{L^6(\mathbb{T}^2)} \le C_N \|u\|_{0,1}$$

where $\mathrm{supp}\, \hat{u}(\cdot, t) \subset [-N, N]$

$$C_N = \exp\left(c\, \frac{\ln N}{\ln \ln N} \right)$$

- initial (nonlinear part of) Hamiltonian

$$H_1(q) = \sum_{n_1 - n_2 + \cdots = 0} q_{n_1} \overline{q}_{n_2} \cdots \overline{q}_{n_6}$$

$$\mathcal{H}_1 = \int_{\mathbb{T}} H_1(q(t))\, dt \ll (n_3^*)^{\varepsilon} \|q\|_{0,1}^6$$

where $q = \sum q_n(t)\, e^{inx}$ $\quad$ $n_i^* = i^{th}$ largest freq.

- induction: Assume that all the Hamiltonians involved in the process satisfy

(7.12) $$\mathcal{H}_1 = \int_{\mathbb{T}} H_1(q(t))\, dt \ll (n_3^*)^{\varepsilon} \cancel{\|q\|_{0,1}^{2r}}$$

$\lesssim 1$ if $\|q\|_{0,1} < C$

$\xRightarrow{\text{duality}}$ $$\left\| \frac{\partial \mathcal{H}_1}{\partial q} \right\|_{0,\infty} \ll (n_3^*)^{\varepsilon}$$

- By direct computation (with (7.12))

$$\left\| \frac{\partial \mathcal{F}}{\partial q} \right\|_{0,1} \ll (\ln n_1^*)(n_3^*)^{\varepsilon}$$

Thus, given

$$\{H_1, F\} = i \sum_n \left[\frac{\partial H_1}{\partial q_n} \frac{\partial F}{\partial \bar{q}_n} - \frac{\partial H_1}{\partial \bar{q}_n} \frac{\partial F}{\partial q_n} \right],$$

we have

$$\left| \int_{\mathbb{T}} \sum_n \frac{\partial \mathcal{H}_1}{\partial \bar{q}_n} \frac{\partial \mathcal{F}}{\partial q_n} \, dt \right| = \left| \left\langle \frac{\partial \mathcal{F}}{\partial q}, \frac{\partial \mathcal{H}_1}{\partial q} \right\rangle_{x,t} \right|$$

$$\leq \left\| \frac{\partial \mathcal{H}_1}{\partial q} \right\|_{0,\infty} \left\| \frac{\partial \mathcal{F}}{\partial q} \right\|_{0,1} \ll (n_3^*)^{\varepsilon} \ln n_1^*$$

$$\left(\begin{array}{lll} \text{If} & n_i \leftrightarrow \{H_1, F\} & \\ & j_i \leftrightarrow H_1 & , \text{ then } n_i^* \geq \max(j_i^*, k_i^*) \\ & k_i \leftrightarrow F & \end{array} \right.$$

" $\Rightarrow$ " (7.12)

- Space-time estimate $\Rightarrow$ spatial estimate

Denote $\quad \tilde{q}_n(t) = q_n e^{in^2 t}$

$$\Rightarrow \|\tilde{q}\|_{0,1} = \|q\|_2 . \quad \text{and} \quad \frac{\partial F}{\partial \bar{q}} = \left. \frac{\partial \mathcal{F}}{\partial \bar{\tilde{q}}} \right|_{t=0}$$

$$\Rightarrow \left\| \frac{\partial F}{\partial \bar{q}} \right\|_2 \leq \left\| \frac{\partial \mathcal{F}}{\partial \bar{\tilde{q}}} \right\|_{L^\infty_t L^2_x} \leq \left\| \frac{\partial \mathcal{F}}{\partial \bar{\tilde{q}}} \right\|_{0,1} \ll 1 .$$

• **Back to I-method:** $H(q) = \sum n^2 |q_n|^2 + N(q)$

$$\frac{d}{dt} H(Iq) = \sum m^2(n)\, n^2 \left(\bar{q}_n \frac{\partial N}{\partial \bar{q}_n} - q_n \frac{\partial N}{\partial q_n} \right)$$

$$+ \sum m(n)\, n^2 \left(q_n \frac{\partial N}{\partial q_n}(Iq) - \bar{q}_n \frac{\partial N}{\partial \bar{q}_n}(Iq) \right)$$

$$+ \sum m(n) \left[\frac{\partial N}{\partial q_n}(Iq) \frac{\partial N}{\partial \bar{q}_n} - \frac{\partial N}{\partial q_n} \frac{\partial N}{\partial \bar{q}_n}(I\bar{q}) \right]$$

$$= (1.9) + (1.10) + (1.11).$$

$\Rightarrow$ $(1.9) + (1.10) = 0$ if $\operatorname{supp} q \subset [-N, N]$

$(1.11) = 0$

$\Rightarrow$ can assume $n_1^* = \max(|n_1|, \cdots, |n_{2r}|) > N$.

Sec 2: Basic estimates

(2.1) $$\sum_{n_1 - n_2 + \cdots - n_{2r} = 0} |c(\bar{n})|\, |q_{n_1}|\, |q_{n_2}| \cdots |q_{n_{2r}}|$$

$$\lesssim \max_{\bar{n}} \left\{ (n_3^*)^{\varepsilon} (\ln n_1^*)^c \min\left((n_3^*)^2, |n_1^2 - n_2^2 + \cdots - n_{2r}^2| \right) \right\} \|q\|_2^{2r}$$

↳ follows from the previous part (Sec 7)

↳ prove by induction: ① prove for $\int |\Phi|^6$

② prove for $N' = \{N, F\}$

• Sec 3: GWP, $s > 1/2$

(3.1) $\|q\|_2 < C$

(3.2) $\|Iq\|_{H^1} \lesssim N^{1-s}$

Assume: $H(q) = \sum n^2 |q_n|^2 + \underset{\text{res.}}{N_0(q)} + \underset{\text{nonres}}{N_1(q)} + N_r$

N_0 : $|D(\bar{n})| < N^{2(1-s)+\varepsilon}$

N_1 : $|D(\bar{n})| \geq N^{2(1-s)+\varepsilon}$

N_r : $\|N_r\| < N^{-C}$, C large

where
$$\|N\| = \sup_{*} \sum c(\bar{n}) |q^{(1)}_{n_1}| \cdots |q^{(2r)}_{n_{2r}}|$$

$\uparrow$

all the factors satisfy (3.1)
all the factors except for at most 2 satisfy (3.2).

and $\|N_0\|, \|N_1\| < N^{2(1-s)}$, $s > 1/2$

Choose F s.t. $\{H_0, F\} = -N_1$

$$\Rightarrow H' = H \circ \Gamma^{-1} = H_0 + N_0 + \cancel{N_1} + N_r \circ \Gamma^{-1}$$
$$+ \cancel{\{H_0, F\}} + \{N_0, F\} + \{N_1, F\}$$
$$+ \text{h.o.t. in } F$$

Lemma:

$$(3.6) \qquad \| \{ H_1, H_2 \} \| \lesssim \| H_1 \| \, \| H_2 \|$$

(we only estimate $\frac{\partial H_1}{\partial q} \frac{\partial H}{\partial \bar{q}}$ i.e no cancellation is used...

① $\| F \| < \frac{\| N_1 \|}{N^{2(1-s)+\varepsilon}} \le N^{-\varepsilon}$

$\Rightarrow$ h.o.t with suff. high degrees are absorbed in N_r.

② G = remaining terms

$$\Rightarrow \| \{ N_1, F \} \| \le N^{-\varepsilon} \| N_1 \|$$

$$\| \{ N_0, F \} \| \le N^{2(1-s)} \frac{\| N_1 \|}{N^{2(1-s)+\varepsilon}} = N^{-\varepsilon} \| N_1 \|$$

Write $G = \overline{N_0} + N_1' =$ resonant + nonresonant.

$$\Rightarrow \| N_1' \| \lesssim N^{-\varepsilon} \| N_1 \|$$

and $N_0' := \overline{N_0} + N_0$

$$\Rightarrow \| N_0' \| \lesssim N^{2(1-s)}$$

- Iterate the process

$$\Rightarrow H = \sum n^2 |q_n|^2 + N_0 + N_r$$

$$\| N_0 \| \lesssim N^{2(1-s)}, \quad \| N_r \| < N^{-C}$$

For this H, estimate $\frac{dH}{dt}(Iq(t))$

$$\Rightarrow \quad \frac{dH}{dt}(Iq(t)) \lesssim N^{4-6s+4\varepsilon}$$

$$\Rightarrow \quad T N^{4-6s+4\varepsilon} \lesssim N^{2-2s}$$

$$\Rightarrow \quad N \gtrsim T^{\frac{1}{4s-2-4\varepsilon}}$$

>0

<u>Thm 1</u>: (NLS) is GWP, $s > 1/2$.

$$\| u(t) \|_{H^s} < N^{1-s} \sim |t|^{\frac{1-s}{4s-2}+}$$

· <u>Improvement for $s < 1/2$</u>

<u>Thm 2</u>: $\exists\, s^* < 1/2$ s.t. (NLS) is GWP, $s > s^*$.

<u>Difficulty</u>: to have $\| N \| \lesssim N^{2(1-s)}$, $s < 1/2$

$$\left(\text{from} \quad \begin{array}{l} \|q\|_2 < C \\ \|Iq\|_{H^1} \lesssim N^{1-s} \end{array} \right.$$

<u>Idea</u>: divide NLS into low ($|m| \leq N_1$) and high ($|m| \geq N_1$)

$\Rightarrow$ apply normal form reduction <u>only</u> on <u>low freq</u> part (smoother.)

· *Initial nonlinearity*

$$N(q) = \sum_{\substack{n_1 - n_2 + \cdots - n_6 = 0 \\ |n_j| \le N_1}} q_{n_1} \overline{q_{n_2}} \cdots \overline{q_{n_6}}$$

$$\Rightarrow \quad \| N \| \lesssim N^{2(1-s)}$$

by choosing $\quad N_1 = N^{\frac{1-s}{1-2s}}$

large

$\Rightarrow$ As before, we can bring H into the form:

$$H = \sum n^2 |q_n|^2 + N_0 + N_r$$

N_0 : $\quad |D(\bar{m})| < N^{2(1-s)+\varepsilon}$

$\qquad \|N_0\| < N^{2(1-s)}$

N_r : $\quad \|N_r\| < N^{-C}$

Improvement:

① When $n_1^* \gtrsim N$ and $n_6^* \lesssim N^{9/10}$,

(5.8) $\quad | N_0^{**}(q) | < N^{2(1-s) - \delta}$

② When $n_1^* \gtrsim N$

(5.9) $\quad \| N_0^* \| < N^{2(1-s) - \delta}$

($\Leftarrow$ improved L^6 Strichartz : Lemma 4.1

$$\int_0^{2\pi/D} \int_{\mathbb{T}} \prod_1^3 | S(t)\phi_j |^2 dx\, dt \lesssim N_1^{-\delta'} \prod_1^3 \|\phi_j\|_2^2$$

if $N_1 > N_3^{1+\delta}$, $D > N_1^{\delta}$, $(N_1 \ge N_2 \ge N_3)$

$$\Rightarrow \left| \frac{d}{dt} H(Iq(t)) \right| < N^{4-6s+4\varepsilon-\delta}$$

$$\Rightarrow \quad T N^{4-6s+4\varepsilon-\delta} \lesssim N^{2-2s}$$

$$\Rightarrow \quad N \gtrsim N(T) = T^{\frac{1}{4s-2+\delta-4\varepsilon}}$$

<u>Lemma 5.23</u>: $\frac{1}{2} > s > s^* = \frac{1}{2} - \frac{\delta}{8}$.

$$\begin{cases} i\, w_t + w_{xx} - \mathbb{P}_{N_1}(u|u|^4) = 0 \\ w|_{t=0} = \mathbb{P}_{N_1} u_0 \end{cases}$$

$$\Longrightarrow \quad \|w(t)\|_{H^s} \lesssim N^{1-s}, \qquad |t| < T.$$

<u>Note</u>: $\left|\frac{d}{dt} H(Iq(t))\right| < N^{2(1-s)-\kappa}$, $|t| < T$

since $T = N^{\kappa}$, κ small.

<u>*High freq part*</u>:

$$\begin{cases} i\, v_t + v_{xx} - 3|w|^4 v - 2|w|^2 w^2 \bar{v} \\ \qquad\qquad + O(|v|^2) - \mathbb{P}^c_{N_1}(w|w|^4) = 0 \\ v|_{t=0} = \phi_1 = \mathbb{P}^c_{N_1} u_0 \end{cases}$$

$$\|\phi_1\|_{H^s} \leq Z$$

$\longleftarrow$ NOT in H^1 (rough)

Define $\zeta(t) = \int_{\mathbb{T}} |w|^4(t)\, dx$

$$\Omega(t) = e^{-3i\int_0^t \zeta(t')\,dt'}.$$

and

$$A = |w|^4 - \zeta$$
$$B = |w|^2 w^2 \bar{\Omega}^2(t).$$

$\Rightarrow$ Letting $V = \bar{\Omega} v$, we have

$$\begin{cases} iV_t + V_{xx} - 3AV - 2B\bar{V} + O(|V|^2) - \bar{\Omega}\,\mathbb{P}^c_{N_1}(w|w|^4) = 0 \\ V|_{t=0} = \phi_1 \end{cases}$$

- Perform careful LWP analysis on $[0, \tau]$

$$\left(\begin{array}{l} \hat{A}(0,t) = 0 \\ \bar{\Omega}^2 \text{ in } B \end{array} \right. \Rightarrow \text{extra smoothing.}$$

$\Rightarrow$ low: $\| \mathbb{P}_{N_1} V \|_{s, \frac{1}{2}+} \lesssim N_1^{\overbrace{\frac{s_1 - s}{4}}^{\leq 0}}$ $\quad s_1 < s < 1/2$, s_1 close to s

high: $\| \mathbb{P}^c_{N_1} (V - e^{it\Delta}\phi_1) \|_{s_1, \frac{1}{2}+} \lesssim N^{-5}$

$= \mathbb{P}^c_{N_1} V - e^{it\Delta}\phi_1$

$=$ high freq nonlinear part

- Put everything together.

$$u_0 = \Phi_0 + \Phi_1$$

$$\Big\downarrow \quad \tau = \text{LWP time}$$

Note the difference from Bourgain '98

$$u(\tau) = \underbrace{\left[w(\tau) + \mathbb{P}^c_{N_1} v(\tau) \right]}_{\substack{\| \\ \text{low } \Psi_0}} + \underbrace{\mathbb{P}^c_{N_1} v(\tau)}_{\substack{\| \\ \text{high } \Psi_1}}$$

$$\|\Psi_1\|_{H^s} < Z + N^{-5} < Z + 1$$

- Given T, need to iterate T/τ steps.

$$\Longrightarrow \text{high}: \quad \frac{T}{\tau} N^{-5} \lesssim N_1^{\overbrace{s - s_1}^{>0}/10}$$

(= upperbound on $2Z$)

$$\text{low}: \quad \underbrace{N^{2(1-s)-K} T}_{\text{low } w} + \underbrace{N_1^{\frac{s_1 - s}{5}} \frac{T}{\tau}}_{\text{on } V} < N^{2(1-s)}$$

$\Longrightarrow$ Need ① $N > T^{1/K}$

② $N^{1-s/1-2s} = N_1 > \left(N^c T \right)^{\frac{10}{s - s_1}}$

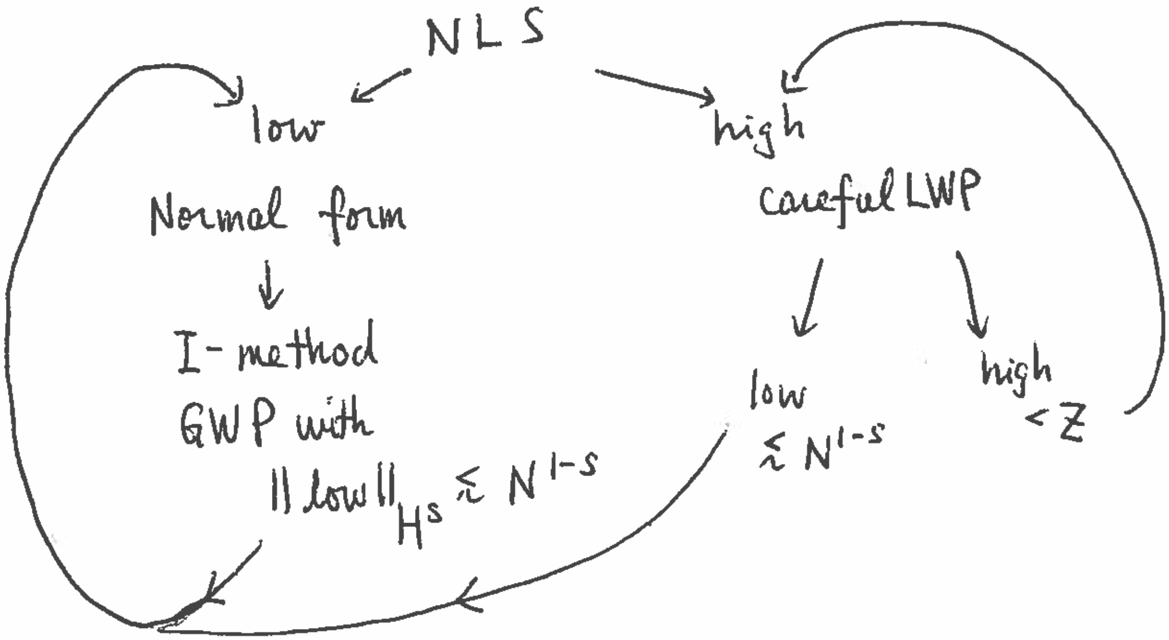
Given T, choose $N > N(T)$ s.t. ① holds

$\Rightarrow$ choose s close to $1/2$ s.t. ② holds.

$$\Rightarrow \quad \| \mathbb{P}_{N_1} u(t) \|_{H^s} \lesssim N^{1-s}$$

$$\| \mathbb{P}_{N_1}^c u(t) \|_{H^s} \lesssim N_1^{\frac{s-s_1}{10}} \sim N^C$$

$$\Rightarrow \quad \| u(t) \| < |t|^{C(s)}$$



- Need high-low separation since normal form reduction does not work for $s < 1/2$.
- low : normal form reduction $\Rightarrow$ I-method
- high : LWP (need extra smoothing estimate.)
- Unlike Bourgain'98, we divide the high freq part

$$v = \underbrace{\mathbb{P}_{N_1} v}_{\text{low}} + \underbrace{\mathbb{P}_{N_1}^c v}_{\text{high}}$$

<u>BO'98:</u> $v = \underbrace{S(t)\Phi_1}_{\text{high}} + \underbrace{w}_{\text{low (smoother)}}$