Hamiltonian Systems and Invariant Measures

Sec 1: Consider a finite dim'l Hamiltonian flow on $\mathbb{R}^{2n}$.

$$(1) \quad \begin{cases} \dot{p}_j = \dfrac{\partial H}{\partial q_j} \\[2ex] \dot{q}_j = -\dfrac{\partial H}{\partial p_j} \end{cases}$$

with Hamiltonian

$$H = H(p_1, \cdots, p_n, q_1, \cdots, q_n)$$

- Recall: Liouville's Theorem: given $\partial_t x = f(x)$, we have $\frac{d}{dt} Vol = div\, f$.

- In particular, the Lebesgue meas $\prod_{j=1}^{n} dp_j\, dq_j$ is invariant under the flow of (1)

$$\left(\because \; div\, X = \sum_{j=1}^{n} \left(\frac{\partial}{\partial p_j} X_j + \frac{\partial}{\partial q_j} X_{n+j} \right) = \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} \frac{\partial H}{\partial q_j} + \frac{\partial}{\partial q_j} \left(- \frac{\partial H}{\partial p_j} \right) \right] = 0 \right.$$

- Also, we have

$$\frac{d}{dt} H(p(t), q(t)) = 0.$$

$\Rightarrow$ Gibbs measures

$$d\mu = Z^{-1} e^{-\beta H(p,q)} \prod_{j=1}^{n} dp_j \, dq_j$$

are <u>invariant</u> under the flow.

Indeed, "$d\mu_f(p,q) = f(H(p,q))\, dp\, dq$"

is invariant under the flow (for reasonable f.)

- <u>Invariance:</u> S^t = flow map.

$$\mu(S^t(p_0, q_0) \in A)$$

$$= \int_A f(\underbrace{H(S^t(p_0,q_0))}_{= H(p_0, q_0)}) \underbrace{dp_t \, dq_t}_{= dp_0 \, dq_0}$$

$$= \mu((p_0, q_0) \in A)$$

- Given an invariant meas, we can view (1) as a dynamical system with meas-preserving transformation.

Poincaré recurrence theorem:

Let h be a dynamical system with phase space M (metric space) and bounded invariant meas. Then, almost all points of M are stable according to Poisson.

Def) M, complete separable metric space.

$h : \mathbb{R} \times M \to M$

(i) $h(0, x) = x, \quad \forall x \in M$

(ii) $h(t, h(s,x)) = h(t+s, x),$

$\forall t, s \in \mathbb{R}, \forall x \in M$

Then, h is called a dynamical system with phase space M.

· positively (negatively) stable according to Poisson if

$\exists \{t_n\} \to \infty$ s.t. $h(t_n, x) \to x$

(or $h(-t_n, x) \to x$)

· Another Version: Fix t, say $t=1$.

Then, (1) with μ and $\underbrace{S^t \text{(with } t=1)}_{=:T}$ is a discrete dynamical system. Then, a version of Poincaré recurrence theorem says:

$\forall A$ with $\mu(A) > 0$,

$\exists N$ s.t. $\mu(T^N A \cap A) > 0$.

'77 · Furstenberg's multiple recurrence theorem

$T: X \to X$, meas-preserving tranf. on a finite meas space $(X, \mathcal{B}, \mu)$.

Given $A \in \mathcal{B}$ with $\mu(A) > 0$ and $k > 0$,

$\exists n > 0$ s.t.

$$\mu(A \cap T^n A \cap T^{2n} A \cap \cdots \cap T^{(k-1)n} A) > 0$$

<u>Aside</u>: Multiple recurrence theorem provides an ergodic theoretic proof of Szemerédi's theorem. Van der Waerden's theorem also has (topological) ergodic theoretic proof.

<u>Sec 2</u>: Hamiltonian formulation of NLS and its finite dimensional approximation.

(NLS) $\quad i u_t - u_{xx} \pm |u|^{p-2} u = 0 \quad$ on $\mathbb{T}$

$$\Longleftrightarrow \quad u_t = i \frac{\partial H}{\partial \bar{u}} \quad \text{with } H(u) = \frac{1}{2}\int |u_x|^2 \pm \frac{1}{p}\int |u|^p$$

(gKdV) $\quad u_t + u_{xxx} \pm u^k u_x = 0$

$$\Longleftrightarrow \quad u_t = \partial_x \frac{\partial H}{\partial u} \quad \text{with } H(u) = \frac{1}{2}\int u_x^2 \mp \frac{1}{(k+1)(k+2)}\int u^{k+2}$$

- A Hamiltonian formulation of (NLS) is often written as

$$u_t = i \frac{\partial \tilde{H}}{\partial \bar{u}} \quad \text{with} \quad \tilde{H}(u) = 2H(u)$$

This is just an issue of how to make sense of $\frac{\partial H}{\partial \bar{u}}$

- For us, what is important is the following:

Let $\begin{cases} p_n = \operatorname{Re} \hat{u}_n \\ q_n = \operatorname{Im} \hat{u}_n \end{cases}$

$$\Rightarrow \quad H = H(p, q) = \frac{1}{2} \sum (-n^2)(p_n^2 + q_n^2) + \text{nonlin}$$

and

$$u_t = -iu_{xx} + \cdots$$

$$\Rightarrow \quad \partial_t \hat{u}_n = i n^2 \hat{u}_n$$

$$\rightarrow \quad \partial_t \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial q_n} \\ -\frac{\partial H}{\partial p_n} \end{pmatrix}$$

(Note: This corresponds to $\partial_t c_n = 2i \frac{\partial H}{\partial \bar{c}_n}$

$$c_n = \hat{u}_n$$

Consider a finite dim'l approximation to (NLS)

$$\text{(FNLS)} \quad u_t^N = -i\, u_{xx}^N \pm i\, \mathbb{P}_N\left(|u^N|^{p-2} u^N\right)$$
$$u^N\big|_{t=0} = u_0^N = \mathbb{P}_N u_0$$

- (FNLS) is a Hamiltonian system with Hamiltonian

$$H_N(\phi) = \frac{1}{2}\int |\mathbb{P}_N \phi_x|^2 \pm \frac{1}{p}\int |\mathbb{P}_N \phi|^p dx$$
$$= H(\mathbb{P}_N \phi)$$

- We can write (FNLS) as a system of $4N+2$ equations : $\partial_t (p, q) = F(p, q)$ where F is a polynomial of degree $p-1$.

$\Rightarrow$ By Cauchy-Lipschitz theorem for ODEs, for every $u_0^N \in E_N = \text{span}\{e^{inx}\}_{n=-N}^{N}$, $\exists!$ local-in-time soln of (FNLS)

Moreover, the solution exists globally in time or $\exists\, T$ s.t.

$$\lim_{t\to T} \max_n \left(|p_n(t)|, |q_n(t)|\right) = \infty$$

- (NLS) preserves the L^2-norm

$\Rightarrow$ (FNLS) also preserves the L^2-norm.

$$\partial_t \int |u^N|^2 dx = 2 \operatorname{Re} \int u^N_t \overline{u^N}$$

$$= 2 \operatorname{Re} \int (\mathbb{P}_N u^N_t)(\overline{\mathbb{P}_N u^N})$$

$\Rightarrow$ (FNLS) has a unique global-in-time solution.

$\Rightarrow$ From Sec 1, the Gibbs meas

$$d\mu_N = Z_N^{-1} e^{-H(p,q)} \prod dp_n \, dq_n$$

$$= Z_N^{-1} e^{\mp \frac{1}{p} \int |u|^p} \prod e^{-\frac{1}{2}(p_n^2 + q_n^2) n^2} dp_n \, dq_n$$

is invariant under the flow of (FNLS).

<u>Q</u>: Is the Gibbs meas

$$\text{"} d\mu = Z^{-1} e^{-H(u)} \prod_{x \in \mathbb{T}} du(x) \text{"}$$

invariant under the flow of (NLS)?

Sec 3: Construction of Gibbs meas

$$d\mu = Z^{-1} e^{-H(u)} \prod_{x \in \mathbb{T}} du(x)$$

$$= Z^{-1} e^{\mp \frac{1}{p} \int |u|^p} \underbrace{e^{-\frac{1}{2}\int |u_x|^2} \prod_{x \in \mathbb{T}} du(x)}_{= d\rho \text{ (unnormalized) Wiener meas}}$$

- There are several ways to make sense of $d\rho$.

In order to avoid an issue at zero mode, we'll consider

$$d\rho = e^{-\frac{1}{2}\int |u|^2 - \frac{1}{2}\int |u_x|^2} \prod_{x \in \mathbb{T}} du(x)$$

(a) First, let's regard $d\rho$ as a limit of

$$d\rho_N = Z_N^{-1} e^{-\sum_{|n| \le N} \frac{1}{2} \langle n \rangle^2 (p_n^2 + q_n^2)} \prod_{|n| \le N} dp_n \, dq_n$$

$\Leftarrow$ prob. meas on $\mathbb{R}^{4N+2}$.

This induces a prob. meas on E_N under the map

$$(p_n, q_n)_{|n| \le N} \longmapsto \sum_{|n| \le N} (p_n + i q_n) e^{inx}.$$

(still denoted by ρ_N)

- We can view ρ_N as the prob distribution of the E_N-valued random variable

$$\omega \longmapsto \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle} e^{inx} = u^N(x)$$

where $\{g_n\}$ = seq of indep standard complex Gaussian r.v's.

- Let $M > N$.

$$\mathbb{E} \| u^M - u^N \|_{H^s}^2 = \mathbb{E} \sum_{N < |n| \leq M} \frac{|g_n|^2}{\langle n \rangle^{2-2s}}$$

$$= \sum_{N < |n| \leq M} \frac{1}{\langle n \rangle^{2-2s}} < \infty$$

or rather $\to 0$

i.e. Cauchy in $L^2(\Omega; H^s)$

$\iff \quad s < 1/2.$

- Indeed, for $s < 1/2$, the map

$$\omega \longmapsto \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx}$$

defines a (Gaussian) meas on $H^s(\mathbb{T})$, denoted by ρ.

(large deviation)

<u>Prop 1:</u> Let $B_K = \{ u : \|u\|_{H^s} \leq K \}$

$s < 1/2$ $\Rightarrow$ $\rho_N (B_K^c \cap E_N) \leq C e^{-cK^2}$

$\rho(B_K^c) \leq C e^{-cK^2}$

<u>Proof)</u> By Chebyshev's ineq,

$$e^{cK^2} \rho_N (B_K^c \cap E_N)$$

$$\leq \int_{E_N} e^{c\|u\|^2_{H^s}} \, d\rho_N(u)$$

$$= \prod_{|n| \leq N} \int_{\mathbb{C}} e^{c \langle n \rangle^{2s-2} |g_n|^2} e^{-\frac{1}{2}|g_n|^2} \frac{dg_n}{2\pi}$$

$$= \prod_{|n| \leq N} \frac{1}{1 - 2c\langle n \rangle^{2s-2}}$$

$$\left(\because \ \mathbb{E}(e^{aX^2}) = (1-2a)^{-1/2}, \quad a < 1/2 \right.$$

$$X \sim N_{\mathbb{R}}(0,1)$$

$$= \prod_{|n| \leq N} \left(1 + \frac{2c\langle n \rangle^{2s-2}}{1 - 2c\langle n \rangle^{2s-2}} \right)$$

$$\lesssim \prod_{n \geq 0} (1 + c' \langle n \rangle^{2s-2}) < \infty$$

for $s < 1/2$.

$\because \prod (1 + a_n)$ converges absolutely

$\Leftrightarrow \sum |a_n|$ converges

And, $\sum \langle n \rangle^{2s-2} < \infty \Leftrightarrow s < 1/2$

Other ways to construct ρ.

(b) Gaussian meas on Hilbert spaces

(Centered) Gaussian meas on $\mathbb{R}^n$

$$d\rho(x) = \frac{1}{\sqrt{(2\pi)^n \det B}} \exp\left(-\frac{1}{2}\langle B^{-1}x, x\rangle_{\mathbb{R}^n}\right)$$

where B = sym, pos, real $n \times n$ matrix.

Zhidkov $\Rightarrow$ extend this definition to infinite dim'l real separable Hilbert spaces.

(c) <u>characteristic function</u>

Simon A Gaussian meas ρ_β on $\mathbb{R}$ with mean 0 and variance β is completely characterized by its char. func.

$$S(\xi) = \int e^{i\langle \xi, x\rangle} \rho_\beta(dx) = e^{-\frac{1}{2\beta}\langle x, x\rangle}$$

(again a Gaussian.)

Given an infinite dim'l Hilbert space H

$\phi \in H$, $\xi \in H^* = H$,

can we construct a meas ρ_β on H s.t.

$$S(\xi) = \int e^{i\langle \phi, \xi\rangle} \rho_\beta(d\phi) = e^{-\frac{1}{2\beta}\langle \xi, \xi\rangle} ?$$

<u>NO</u>:

Let $\{e_n\}$ be an O.N. basis of H.

Then, $\quad S(e_n) = e^{-\frac{1}{2\beta}}$

But for any $\phi \in H$, we have $\langle \phi, e_n\rangle \to 0$ as $n \to \infty$

$\Rightarrow$ By DCT, $\lim\limits_{n\to\infty} S(e_n) = 1 \neq e^{-\frac{1}{2\beta}} \Rightarrow$ X

- need to construct such meas on larger spaces of distributions (by Bocher-Minlos theorem)

  - works if we regard $\langle \phi, \xi\rangle$ as $\mathcal{S}-\mathcal{S}'$ duality

(d) Abstract Wiener spaces

Gross
Kuo

H = real separable Hilbert space

$\mathcal{F}$ = { finite dim'l orth. projections $\mathbb{P}$ }

$\Rightarrow$ Given $\mathbb{P} \in \mathcal{F}$ and F = Borel subset of $\mathbb{P}H$, define a cylinder set E by

$$E = \{ x \in H : \mathbb{P}x \in F \}$$

Let $\mathcal{R}$ = { all cylinder sets }

$\mathcal{R}$ is a field but NOT a σ-field.

Define Gauss meas on H:

$$\rho(E) = (2\pi)^{-n/2} \int_F e^{-\|x\|^2/2} \, dx$$

where $E \in \mathcal{R}$
$n = \dim \mathbb{P}H$
dx = Lebesgue meas on $\mathbb{P}H$.

<u>Note</u>: ρ is finitely additive but <u>NOT</u> countably additive

<u>Idea</u>: Extend ρ onto a larger Banach space B

$$H \hookrightarrow B$$

- <u>Seminorm $\|\cdot\|_B$ on H</u>:

$\|\cdot\|_B$ is called <u>measurable</u> if $\forall\, \varepsilon > 0$

$\exists\, \mathbb{P}_\varepsilon \in \mathcal{F}$ s.t.

$$\rho\left(\| \mathbb{P} x \|_B > \varepsilon \right) < \varepsilon$$

for all $\mathbb{P} \in \mathcal{F}$ orthogonal to $\mathbb{P}_\varepsilon$.

<u>Rmk</u>: • $\|\cdot\|_B$ meas is weaker than $\|\cdot\|_H$

• H is NOT complete under meas $\|\cdot\|_B$ unless $\dim H < \infty$

$\Rightarrow$ Let B = completion of H under $\|\cdot\|_B$.

i = inclusion map $i : H \hookrightarrow B$.

<u>Thm</u> (Gross)

ρ can be extended to $\tilde{\rho}$ on B

s.t. $\tilde{\rho}$ is countably additive on B

For simplicity, we use ρ for $\tilde{\rho}$.

Then, (i, H, B) or (B, ρ) is called an abstract Wiener space.

ex) $H = H^1$

i.e. we consider $d\rho = Z^{-1} e^{-\frac{1}{2}\int |u|^2 + |u_x|^2} \prod_{x \in \mathbb{T}} du(x)$

- $B_1 = H^s, \quad s < 1/2$
- $B_2 = \mathcal{F}L^{s,\infty}, \quad s < 1$

where $\|f\|_{\mathcal{F}L^{s,\infty}} = \sup_n \langle n \rangle^s |\hat{f}(n)|$

- $B_3 = \mathcal{F}L^{s,p}, \quad (s-1)p < -1$
  $p < \infty$

where $\|f\|_{\mathcal{F}L^{s,p}} = \| \langle n \rangle^s |\hat{f}(n)| \|_{\ell^p_n}$

- $B_4 = B^{1/2}_{p,\infty}, \quad p < \infty$

etc.

idea: For example,

$$\mathbb{E} \|f\|^p_{\mathcal{F}L^{s,p}} = \mathbb{E} \sum_n \langle n \rangle^{sp} \frac{|g_n(\omega)|^p}{\langle n \rangle^p}$$

$$\sim \sum_n \langle n \rangle^{(s-1)p} < \infty$$

$$\Longleftrightarrow (s-1)p < -1$$

<u>Thm</u> (Fernique's integrability theorem)

$$\int_B e^{c\|x\|_B^2} \rho(dx) < \infty.$$

This theorem, in particular, implies the large deviation lemma.

$$\mathbb{P}\left(\|u\|_B > K \right) < C e^{-cK^2}$$

as in Prop 1.

Now, back to

$$d\mu = Z^{-1} e^{\mp \frac{1}{p} \int |u|^p dx} \underbrace{e^{-\frac{1}{2} \int |u_x|^2 dx} \prod_{x \in \mathbb{T}} du(x)}_{= d\rho.}$$

- <u>defocusing case</u> : μ is a prob meas on $\bigcap_{s < 1/2} H^s$. and $\mu \ll \rho$.
- If (NLS) is focusing, $\frac{1}{p} \int |u|^p dx$ is unbounded

Moreover,

For $p > 2$, by Hölder, we have

$$\int_{\mathbb{T}} |u|^p \geq \|u\|_{L^2}^p = \left(\sum |\widehat{u}_n|^2\right)^{p/2}$$

$$= \left(\sum \left|\frac{g_n}{\langle n\rangle}\right|^{p\cdot 2/p}\right)^{p/2}$$

$$\geq \sum \left|\frac{g_n}{n}\right|^p \qquad (\because\ \ell^1 \supset \ell^{(2/p)}) \quad {}^{<1}$$

$$\Rightarrow \mathbb{E}_\rho\left(e^{\int |u|^p}\right) \geq \prod_n \mathbb{E}\left(e^{|g_n|^p/\langle n\rangle^p}\right) = \infty.$$

$$\left(\because\ \mathbb{E}(e^{|g|^p}) = \frac{1}{2\pi}\int e^{|g|^p} e^{-|g|^2/2}\, dg \right.$$

$$= \infty \text{ for } p > 2.$$

Recall that L^2-norm is conserved.

$\Rightarrow$ Introduce an L^2-cutoff in μ.

<u>Thm</u> (Lebowitz - Rose - Speer '88, Bourgain '94)

① $p < 6$:

$$\circledast \qquad e^{\int |u|^p} \chi_{\{\|u\|_{L^2} \leq B\}} \in L^r(d\rho)$$

$\forall\ r, B < \infty$

In particular, $\mu \ll \rho$

(2) $p = 6$

⊛ holds for $B < B_r$

(3) $p > 6$: ⊛ fails.

Rmk: $p = 6$ is related to the L^2-criticality of quintic NLS / gKdV.

Proof of (1) and (2):

Recall $\hat{u}_n = \frac{g_n}{\langle n \rangle}$ on supp ρ.

First, estimate $\rho\left(\|u\|_{L^p} > K,\ \|u\|_{L^2} \leq B \right)$

- By Sobolev ineq,

$$\| \mathbb{P}_{\leq M_0} u \|_{L^p} \leq C M_0^{\frac{1}{2} - \frac{1}{p}} \underbrace{\left(\sum_{|n| \leq M_0} |\hat{u}_n|^2 \right)^{1/2}}_{\leq B}$$

Choose M_0 dyadic s.t.

$$\frac{1}{2} K = C M_0^{\frac{1}{2} - \frac{1}{p}} B$$

i.e. $$M_0 \sim \left(\frac{K}{B} \right)^{\frac{1}{\frac{1}{2} - \frac{1}{p}}}$$

Let $M_j = M_0 2^j$.

Then, we have

$$\| \mathbb{P}_{M_j} u \|_{L^p} \le c M_j^{\frac{1}{2}-\frac{1}{p}} \| \mathbb{P}_{M_j} u \|_{L^2}$$

- Let $\{\sigma_j\}$ s.t. $\sum \sigma_j = \frac{1}{2}$

In particular, take $\sigma_j = C 2^{-\varepsilon j} = C M_0^{\varepsilon} M_j^{-\varepsilon}$.

Then, we have

$$P\left(\|u\|_{L^p} > K, \ \|u\|_{L^2} \le B \right)$$

$$\le \sum_{j=1}^{\infty} \underbrace{P\left(\| \mathbb{P}_{M_j} u \|_{L^p} > \sigma_j K \right)}$$

$$\le P\left(\| \mathbb{P}_{M_j} u \|_{L^2} > c\, \sigma_j M_j^{\frac{1}{p}-\frac{1}{2}} K \right)$$

$$\sim P\left(\Big(\sum_{|n| \sim M_j} |g_n(\omega)|^2 \Big)^{1/2} \gtrsim \underbrace{\sigma_j M_j^{\frac{1}{p}+\frac{1}{2}} K} \right)$$

<u>Note</u>: $R_j \sim \sigma_j M_j^{\frac{1}{p}+\frac{1}{2}} K \sim M_0^{\varepsilon} M_j^{\frac{1}{2}+\frac{1}{p}-\varepsilon} K =: R_j$.

$$\gtrsim M_0^{\varepsilon} M_j^{\frac{1}{2}+}.$$

<u>Lemma 2</u>: $\{\tilde{g}_n\}$, indep standard real-valued Gaussian r.v's. $\forall M \geq 1$,

$$P\left[\left(\sum_{n=1}^{M} \tilde{g}_n^2\right)^{1/2} \geq R\right] \leq e^{-\frac{1}{4}R^2}, \quad R \geq 3M^{1/2}$$

—

$$\Rightarrow \rho\left(\|u\|_{L^p} > K, \ \|u\|_{L^2} \leq B\right)$$

$$\leq \sum_{j=1}^{\infty} e^{-cR_j^2}$$

$$R_j^2 = \sigma_j^2 M_j^{\frac{2}{p}+1} K^2, \quad \sigma_j^2 = \underline{\underline{2^{-2\varepsilon j}}}, \quad M_j^{\frac{2}{p}+1} = M_0^{\frac{2}{p}+1} \underline{\underline{2^{(\frac{2}{p}+1)j}}}$$

$$\sim e^{-cK^2 M_0^{\frac{2}{p}+1}}$$

$$\sim e^{-cK^{2+\frac{\frac{2}{p}+1}{\frac{1}{2}-\frac{1}{p}}} B^{-\frac{\frac{2}{p}+1}{\frac{1}{2}-\frac{1}{p}}}}$$

$$2+\frac{\frac{2}{p}+1}{\frac{1}{2}-\frac{1}{p}} = \frac{4p}{p-2}, \qquad -\frac{\frac{2}{p}+1}{\frac{1}{2}-\frac{1}{p}} = -\frac{2p+4}{p-2}$$

- $\frac{4p}{p-2} > p$ for $p < 6$.

When $p = 6$, choose B small s.t.

$$cK^6 B^{-4} > K^6 \quad \boxtimes$$

Sec 4: Invariance of Gibbs meas. (Part 1)

Assume that GWP of a given PDE is known in $\bigcap_{s<1/2} H^s$.

ex) $iu_t - u_{xx} \pm |u|^{p-2}u = 0, \quad p \leq 4$

KdV, etc.

Lemma (Uniform approximation lemma)

Let $u_0 \in H^s$. Then, $\forall\, T < \infty$, $s_1 < s$,

$$\| u - u^N \|_{C([-T,T]; H^{s_1})} \to 0$$

as $N \to \infty$.

Idea of proof) Compare u and u^N in Duhamel form

$$u(t) - u^N(t) = S(t)u_0 - S(t)u_0^N \pm i\left(\int_0^t S(t-t')\left(|u|^2u - \mathbb{P}_N(|u^N|^2u^N)\right)(t')dt'\right.$$

- $\underbrace{\| u_0 - u_0^N \|_{H^{s_1}}}_{\text{only on high freq}} \lesssim N^{s_1 - s} \| u_0 - u_0^N \|_{H^s}$

$$\lesssim N^{s_1 - s} \| u_0 \|_{H^s} \to 0.$$

- Use

$$\mathbb{P}_N \left((\mathbb{P}_{N/3} v)^3 \right) = (\mathbb{P}_{N/3} v)^3$$

$$\Rightarrow |u|^2 u - \mathbb{P}_N (|u^N|^2 u^N)$$

$$= |u|^2 u - \mathbb{P}_N (|u|^2 u)$$

$$+ \mathbb{P}_N (|u|^2 u - |u^N|^2 u^N)$$

$$|u|^2 u - |\mathbb{P}_{N/3} u|^2 \mathbb{P}_{N/3} u$$

$$+ \mathbb{P}_N \left(|\mathbb{P}_{N/3} u|^2 \mathbb{P}_{N/3} u - |u|^2 u \right)$$

Hence, in estimating these terms, we have

$u - \mathbb{P}_{N/3} u$ or $u - u^N$

can hide in LHS.

Let S^t and S_N^t be soln maps of (NLS) and (FNLS). Then, by Lemma, we have

$$(2) \qquad S^t \phi = \lim_{N \to \infty} S_N^t (\mathbb{P}_N \phi) \text{ in } H^{s_1}$$

for $\phi \in H^s$, $s_1 < s$.

<u>Invariance:</u> $s_i < 1/2$

K, compact in $\bigcap_{s<1/2} H^s$

(3)
$$\mu\left(S^t K + \overline{B_\varepsilon} \right) \geq \overline{\lim}\, \mu_N \left(\left(S^t K + \overline{B_\varepsilon} \right) \cap E_N \right)$$

(4) By (2), $S_N^t(\mathbb{P}_N K) \subset S^t(K) + B_{\varepsilon/2}$.

Then, $\forall \varepsilon\ \exists \delta$ s.t.

$$S_N^t\left((K + B_\delta) \cap E_N \right)$$

$$\underset{\text{local theory}}{\subset} S_N^t(\mathbb{P}_N K) + B_{\varepsilon/2} \qquad t \text{ small \& fixed}$$

$$\underset{(4)}{\subset} S^t K + B_\varepsilon$$

By invariance of μ_N,

(5)
$$\mu_N\left((K + B_\delta) \cap E_N \right) = \mu_N\left(S_N^t\left((K + B_\delta) \cap E_N \right) \right) \leq \mu_N\left(\left(S^t K + B_\varepsilon \right) \cap E_N \right)$$

$$
\begin{aligned}
\mu(K) &\le \mu(K + B_\delta) \\
&\le \underline{\lim}\ \mu_N\big((K + B_\delta) \cap E_N\big) \\
&\overset{(5)}{\le} \underline{\lim}\ \mu_N\big((S^t K + B_\varepsilon) \cap E_N\big) \\
&\le \overline{\lim}\qquad \vdots \\
&\le \overline{\lim}\ \mu_N\big((S^t K + \overline{B}_\varepsilon) \cap E_N\big) \\
&\underset{(3)}{\le} \mu(S^t K + \overline{B}_\varepsilon)
\end{aligned}
$$

$\Rightarrow$ Let $\varepsilon \to 0$. $\mu(K) = \mu(S^t K)$

By time-reversibility,

$$\mu(K) = \mu(S^t K).$$

Sec 5: Invariance of Gibbs meas (Part 2)

Almost sure global well-posedness

Now, we consider the situation where we do **not** have a priori GWP.

<u>main idea</u>: use invariant Gibbs meas as a replacement of conservation laws

$$\begin{tikzcd} \text{GWP} \arrow[d, bend left] \\ \text{invariance} \arrow[u, bend left] \end{tikzcd}$$

In reality, we use almost sure GWP of (FNLS) (with estimates) and invariance of μ_N.

$$\begin{tikzcd} \mu_N \arrow[r, "\text{weak convergence}"] \arrow[d, leftrightarrow, "\text{invariance}"'] & \mu \arrow[d, dotted, dash] \\ u^N \arrow[r, "\text{unif convergence}"'] & u \end{tikzcd}$$

- "Almost" a.s. GWP $\Rightarrow$ a.s. GWP.

$\forall\, T, \varepsilon > 0$. $\exists\, \Omega_{T,\varepsilon}$ s.t

(i) $\mathbb{P}(\Omega^c_{T,\varepsilon}) < \varepsilon$

(ii) For each $\omega \in \Omega_{T,\varepsilon}$, $\exists!$ soln u with $u|_{t=0} = u_0(\omega)$

For fixed $\varepsilon > 0$, let

$$T_j = 2^j \text{ and } \varepsilon_j = \varepsilon / 2^j$$

$\Rightarrow$ construct $\Omega_{T_j, \varepsilon_j}$

$\Rightarrow$ Let $\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \Omega_{T_j, \varepsilon_j}$

Then, we have GWP for $\omega \in \Omega_\varepsilon$ and $\mathbb{P}(\Omega_\varepsilon^c) < \varepsilon$.

Now, let $\tilde{\Omega} = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$. Then, $\mathbb{P}(\tilde{\Omega}^c) = 0$.

<u>Prop 3</u>: $T < \infty$, $\varepsilon > 0$

$\exists\, \Omega_N = \Omega_N(\varepsilon, T)$ s.t.

(i) $\mu_N(\Omega_N^c) < \varepsilon$

(ii) For $u_0^N \in \Omega_N$, $\exists!$ soln u^N of (FNLS)

$s < 1/2$ s.t.

$$\| u^N(t) \|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad |t| \le T$$

<u>Proof</u>) By local theory, we have

$$\| u_0^N \|_{H^s} \le K \Rightarrow \| u^N(t) \|_{H^s} \le 2K$$

for $|t| \le \delta \sim K^{-\theta}$, $\theta > 0$.

Let
$$\Omega_N = \bigcap_{j=-[T/\delta]}^{[T/\delta]} S_N^{j\delta}\left(\{\|u_0^N\|_{H^s} \le K\}\right)$$

- By invariance of μ_N, we have

$$\mu_N(\Omega_N^c) \lesssim \frac{T}{\delta}\,\mu_N\left(\{\|u_0^N\|_{H^s} > K\}\right)$$
$$\lesssim T K^\theta e^{-cK^2}$$

$\Rightarrow$ Choose $K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$. $\Rightarrow \mu_N(\Omega_N^c) < \varepsilon$

- By its construction,

$$\|u^N(j\delta)\|_{H^s} \le K, \quad j = 0, \cdots, \pm[T/\delta]$$

- By local theory,

$$\|u^N(t)\|_{H^s} \le 2K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad |t| \le T$$

$\square$

Lemma 4 (Approximation lemma): $s < 1/2$.

Let $u_0 \in H^s$ s.t. $\|u_0\|_{H^s} \le K$.

Assume that the soln u^N of (FNLS) with initial condition $u_0^N = \mathbb{P}_N(u_0)$ satisfies the bound

$$\|u^N(t)\|_{H^s} \le K, \qquad |t| \le T.$$

Then, $\exists!$ soln u of (NLS) with $u|_{t=0} = u_0$ on $[-T, T]$. Moreover, $\exists$ C_0, C_1, C_2 s.t.

$$\|u(t) - u^N(t)\|_{H^{s_1}} \le C_0 e^{C_1 (1+K)^{C_2} T} K \underbrace{N^{s_1 - s}}_{\downarrow \, 0}$$

for $s_1 < s$.

Sketch of proof: Note that (NLS) with $u|_{t=0} = u_0$ and (FNLS) with $u^N|_{t=0} = u_0^N$ are both well-posed on $[-\delta, \delta]$, $\delta \sim (1+K)^{-\theta}$, indep of N.

· By computing the difference $u - u^N$ in the $X^{s_1, b}$-norm, we have

$\| u - u^N \|_{X^{s_1, b}}$

$\lesssim K N^{s_1 - s}$ + nonlinear terms

$\uparrow$

contain either $u - \mathbb{P}_{N/3} u$ or $u - u^N$

- $\|$ terms with $u - \mathbb{P}_{N/3} u \|_{X^{s_1,b}}$

$\lesssim \| u \|^{p-2}_{X^{s_1,b}} \cdot \| u - \mathbb{P}_{N/3} u \|_{X^{s_1,b}}$

$\lesssim K^{p-1} N^{s_1 - s}$

δ^θ from smallness of the time interval

- $\|$ terms with $u - u^N \|_{X^{s_1,b}}$

$\lesssim \left(\| u \|^{p-2}_{X^{s_1,b}} + \| u^N \|^{p-2}_{X^{s_1,b}} \right) \| u - u^N \|_{X^{s_1,b}}$

$\lesssim \delta^\theta K^{p-2} \| u - u^N \|_{X^{s_1,b}}$

$\Longrightarrow \quad \| u - u^N \|_{X^{s_1,b}} \le C K N^{s_1 - s} + \frac{1}{2} \| u - u^N \|_{X^{s_1,b}}$

$\Rightarrow \quad \| u - u^N \|_{X^{s_1,b}} \le 2 C K N^{s_1 - s}$

$$\Rightarrow \quad \| u(t) - u^N(t) \|_{H^{s_1}} \le 2C' K N^{s_1 - s}$$

2^{nd} time ($|t| \le 2\delta$)

$$\le 2 \cdot 2C' K N^{s_1 - s}$$

j^{th} time ($|t| \le j\delta$)

$$\le 2^j \cdot 2C' K N^{s_1 - s}$$

$\Rightarrow$ Repeating $\sim \frac{T}{\delta}$ many times, we obtain

$$\| u(t) - u^N(t) \|_{H^{s_1}} \lesssim e^{\frac{T}{\delta}} K N^{s_1 - s}$$

$$e^{\frac{T}{\delta}} = e^{C_1 (1+K)^{C_2} T}$$

$\boxtimes$

<u>Prop 5</u> ("Almost" a.s. GWP) : $s < 1/2$

Given $T, \varepsilon > 0$, $\exists\, \Omega(\varepsilon, T)$

(i) $\mu(\Omega^c(\varepsilon, T)) < \varepsilon$

(ii) For $u_0 \in \Omega(\varepsilon, T)$, $\exists !$ soln u of (NLS)

s.t.

$$\| u(t) \|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad |t| \le T$$

<u>Proof)</u> Let $\tilde{\Omega}_N = \tilde{\Omega}_N(\varepsilon, T)$ be the set constructed in Prop. 3. Recall $K \sim \left(\log \frac{T}{\varepsilon} \right)^{1/2}$

Let $\Omega_N = \Omega_N(\varepsilon, T)$

$$= \{ u_0 \in H^s ; \ \|u_0\| \le K , \ \mathbb{P}_N u_0 \in \tilde{\Omega}_N \}.$$

Then, from Prop 3, we have

$$\| \Phi_N(t)(\mathbb{P}_N u_0) \|_{H^s} \le 2K , \quad |t| \le T$$

for $u_0 \in \Omega_N$

- By Lemma 4, $\exists N_1$ s.t.

$$\| u(t) - u^N(t) \|_{H^{s_1}} \ll 1 , \quad |t| \le T$$

for $N \ge N_1$.

$$\Rightarrow \| u(t) \|_{H^{s_1}} \lesssim \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad |t| \le T.$$

Now, we estimate $\mu(\Omega_N^c)$.

$$\Omega_N^c \subset \underbrace{\{ \|u_0\|_{H^s} \ge K \}}_{= A} \cup \underbrace{\{ \mathbb{P}_N u_0 \in \tilde{\Omega}_N^c \}}_{B_N}$$

- $\mu(A) < \varepsilon$
- $B_N \cap E_N = \tilde{\Omega}_N^c$.

$$\Rightarrow \rho(B_N) = \rho_N(\tilde{\Omega}_N^c)$$

Recall $\mu \ll \rho$.

Moreover, $d\tilde{\rho}_N := \chi_{\{\|u^N\|_{L^2} \le B\}} \, d\rho_N$

$\ll d\mu_N$. $\qquad d\tilde{\rho}_N = \tilde{R}_N \, d\mu_N$

$$\to \quad \mu(B_N) = \int_{B_N} G \, d\rho$$

$$\le \left(\int G^2 d\rho\right)^{1/2} \left(\int_{\tilde{\Omega}_N^c} d\tilde{\rho}_N\right)^{\frac{1}{2}}$$

$$\lesssim \underbrace{\left(\int \tilde{R}_N^2 \, d\mu_N\right)^{\frac{1}{4}}}_{= \left(\int \tilde{R}_N \, d\tilde{\rho}_N\right)^{\frac{1}{4}} \le C \text{ indep of } N} \mu_N(\tilde{\Omega}_N^c)^{\frac{1}{4}}$$

$$\lesssim \mu_N(\tilde{\Omega}_N^c)^{\frac{1}{4}} < \varepsilon$$

$\boxtimes$

- Invariance of Gibbs measure

Let $X = \bigcup_N \{ F = F(\hat{u}_{-N}, \dots, \hat{u}_N),$ bounded & conti $\}$.

- Fix $F \in X$.

$\mu_N \rightharpoonup \mu$

$$\Rightarrow \left|\int F \, d\mu - \int F \, d\mu_N\right|$$

$$+ \left|\int F \circ \Phi(t) \, d\mu - \int F \circ \Phi(t) \, d\mu_N\right| < \varepsilon, \quad N \ge N_1$$

$\cdot \; \left| \int F \circ \Phi(t)\, d\mu_N - \int F \circ \Phi_N(t)\, d\mu_N \right|$

$$\leq 2 \, \|F\|_{L^\infty} \, e^{-cK^2} + \varepsilon$$

on $\|u^N\| \geq K$ $\qquad\qquad$ $\uparrow$ Lemma 4 & continuity of F.

$\lesssim \varepsilon, \quad N \geq N_2$

(ii) $\cdot \; \left| \int F \circ \Phi_N(t)\, d\mu_N - \int F \, d\mu_N \right| = 0$

by invariance of μ_N.

$\Rightarrow \; \left| \int F \circ \Phi(t)\, d\mu - \int F \, d\mu \right| \lesssim \varepsilon, \quad \forall \varepsilon.$

$\Rightarrow \; \int F \circ \Phi(t)\, d\mu = \int F \, d\mu.$

· Further Results

- $H^s \cap \mathcal{F}L^{s+\frac{1}{2},\infty}$, $s < \frac{1}{2}$.
  mKdV (Bourgain '94), Zakharov system (Bo, '94),
  KdV system (Oh, '09), Schrödinger-Benjamin-Ono (Oh, '09)

- variant of H^s: NLS on $\mathbb{D}^2$ (Tzvetkov, '06, '08)
  3-d wave (Burg - Tzvet, '07, '08)

- $\mathcal{H}^s$ with $x^2 - \partial_x^2$: 1-d NLS on $\mathbb{R}$ (Burq - Thomann - Tzvet, '10)
  └ basis of Hermite functions.

- "Almost" invariant finite dimensional meas
  $\Rightarrow$ invariant meas for DNLS
  Nahmod - Oh - Rey-bellet - Staffilani '10

  (i.e. finite dim'l energy is not conserved
  $\rightarrow$ finite dim'l weighted Wiener meas is NOT invariant
  i.e. (6) fails.

- White noise

$$d\mu = Z^{-1} \exp\left(-\frac{1}{2}\int u^2\, dx\right) \prod_{x \in \mathbb{T}} du(x) \quad \text{on } H^{-\frac{1}{2}-\varepsilon}$$

$$\Leftrightarrow \quad u = \sum g_n(\omega) e^{inx}.$$

Quastel - Valkó '08 (by complete integrability relating to mKdV

KdV

Oh '09 (direct proof via 2nd iteration) Note the error in nonlin analysis

OQV '10
" interpolation of white noise and Gibbs meas (invariant)
weakly converges to white noise"

Result of OQV also applies to cubic NLS & mKdV.

⇒ <u>conjecture</u>: white noise is invariant for cubic NLS & mKdV.

Partial result in this direction (cubic NLS) Colliander - Oh '10