On some problems in pointwise ergodic theory

Mariusz Mirek
Universität Bonn, Germany
mirek@math.uni-bonn.de

Abstract

Let $\mathbf{P}$ denote the set of prime numbers. We are going to prove $\ell^r(\mathbb{Z})$–boundedness ($r > 1$) of the discrete maximal function

$$M_{\mathbf{P}_h} f(x) = \sup_{N \in \mathbb{N}} \frac{1}{|\mathbf{P}_h \cap [1, N]|} \Big| \sum_{p \in \mathbf{P}_h \cap [1,N]} f(x - W(p)) \Big|, \quad \text{for} \quad x \in \mathbb{Z},$$

where $W : \mathbb{Z} \mapsto \mathbb{Z}$ is a fixed polynomial and

$$\mathbf{P}_h = \{p \in \mathbf{P} : \exists_{n \in \mathbb{N}}\ p = [h(n)]\},$$

for an appropriate function h.

For instance, one can think that the function h has the following form

$$h_0(x) = x^c, \quad h_1(x) = x \log^5 x, \quad h_2(x) = \frac{x^c}{e^{\log^{1/3} x}}, \quad h_3(x) = x \log \log x,$$

where $c \in (1, 2)$.

As a consequence we obtain related pointwise ergodic theorems along the set $\mathbf{P}_h$.

We also provide some tools necessary to solve the ternary Goldbach problem in the primes $p_1 \in \mathbf{P}_{h_1}, p_2 \in \mathbf{P}_{h_2}, p_3 \in \mathbf{P}_{h_3}$ for various functions h_1, h_2, h_3.

Finally we show that every subset of $\mathbf{P}_h$ having positive relative upper density contains a nontrivial three–term arithmetic progression. In particular the set of Piatetski–Shapiro primes of fixed type $71/72 < \gamma < 1$, i.e. $\{p \in \mathbf{P} : \exists_{n \in \mathbb{N}}\ p = \lfloor n^{1/\gamma} \rfloor\}$ has this feature. We show this by proving the counterpart of Bourgain–Green's restriction theorem for the set $\mathbf{P}_h$.