NERVES OF ALGEBRAS

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INTERESTED IN...

Monads $T = (T, \mu, \eta)$ on cats $\mathcal{B}$, where:

- $\mathcal{B} \simeq [\mathbb{B}^{op}, \mathbf{Set}]$ (some small $\mathbb{B}$)
- T preserves connected limits
- μ and η are cartesian.

E.g.

- $\mathcal{B} = \mathbf{Set}$, T = theory of monoids
- $\mathcal{B} = \mathbf{Set}$, T = theory of M-sets (some monoid M)
- $\mathcal{B} = \mathbf{DirGph}$, T = theory of cats
- $\mathcal{B} = n\text{-}\mathbf{Gph}$, T = theory of strict n-cats
- $\mathcal{B} = n\text{-}\mathbf{Gph}$, T = theory of weak n-cats

REPRESENTABLE FUNCTORS

Let $\mathcal{A}$ be a cat such that every limit-preserving functor out of $\mathcal{A}$ has a left adjoint. E.g.: $\mathcal{A} \simeq [\mathbb{A}^{op}, \mathbf{Set}]$.

Prop'n: For $T: \mathcal{A} \to \mathbf{Set}$,

T is representable

$\Leftrightarrow$ T preserves limits.

FAMILIALLY REPRESENTABLE FUNCTORS (I)

(Diers, Carboni - Johnstone)

$T : \mathcal{A} \longrightarrow \mathbf{Set}$ is familially representable if $T = \sum_{i \in I} \mathcal{A}(T_i, -)$ for some family $(T_i)_{i \in I}$ of objects of $\mathcal{A}$.

Prop'n: For $T : \mathcal{A} \longrightarrow \mathbf{Set}$,

T is familially representable

$\Leftrightarrow$ T preserves connected limits.

FAMILIALLY REPRESENTABLE FUNCTORS (II)

$T : \mathcal{A} \longrightarrow \hat{\mathbb{B}} := [\mathbb{B}^{op}, \mathbf{Set}]$ is familially representable if for all $b \in \mathbb{B}$,

$$T(-)(b) : \mathcal{A} \longrightarrow \mathbf{Set}$$

is familially representable. So

$$(TX)(b) = \sum_{i \in I_b} \hat{\mathbb{B}}(T_{b,i}, X)$$

$(X \in \mathcal{A},\ b \in \mathbb{B})$.

Prop'n: For $T : \mathcal{A} \longrightarrow \hat{\mathbb{B}}$,

T is familially representable

$\Leftrightarrow$ T preserves connected limits.

FAMILIALLY REPRESENTABLE MONADS

A monad (T, μ, η) on $\hat{\mathbb{B}}$ is familially representable if T is familially representable and μ and η are cartesian.

Can say explicitly (in terms of representing families):

- what μ and η are
- what "cartesian" means
- what the monad axioms are
- what T-algebras are.

GENERALIZATION OF $\mathbb{\Delta}$

Let T be a fam. rep. monad on $\widehat{\mathbb{B}}$.

Define a cat $\mathbb{\Delta}_T$ by

- $\mathrm{ob}\, \mathbb{\Delta}_T = \{ (b, i) \mid b \in \mathbb{B},\ i \in Ib \}$
- $$\begin{aligned} \mathbb{\Delta}_T((b,i), (c,j)) &= \widehat{\mathbb{B}}^T(FT_{b,i}, FT_{c,j}) \\ &= \widehat{\mathbb{B}}(T_{b,i}, T(T_{c,j})). \end{aligned}$$

E.g. $\mathbb{B} = (\bullet \rightrightarrows \bullet)$, T = free cat: $\mathbb{\Delta}_T \simeq \mathbb{\Delta}$.

Have

$$\begin{array}{ccc} \mathbb{\Delta}_T & \hookrightarrow & \widehat{\mathbb{B}}^T \\ (b,i) & \longmapsto & FT_{b,i} \end{array}$$

hence nerve functor

$$N : \widehat{\mathbb{B}}^T \longrightarrow \widehat{\mathbb{\Delta}_T}$$

where

$$(NX)(b,i) = \widehat{\mathbb{B}}(T_{b,i}, X).$$

E.g. T = free cat: N is standard nerve functor.

NERVES OF ALGEBRAS

Let T be a fam. rep. monad on $\hat{\mathbb{B}}$.

Thm: $N : \hat{\mathbb{B}}^T \longrightarrow \widehat{\Delta_T}$ defines an equivalence between $\hat{\mathbb{B}}^T$ and the full subcat

$$\{ Z : \Delta_T^{op} \longrightarrow \mathrm{Set} \mid Z \text{ preserves certain limits} \}$$

of $\widehat{\Delta_T}$.

(Those limits: one for each $b \in \mathbb{B}$, $i \in Ib$, $\gamma \in \hat{\mathbb{B}}(T_{b,i}, I)$, of shape $\mathrm{Elt}(T_{b,i})$.)

E.g.

- $\mathbb{B} = (\bullet \rightrightarrows \bullet)$, T = free cat: condition is

$$Z_{n_1 + \cdots + n_k} \xrightarrow{\sim} Z_{n_1} \times_{Z_0} \cdots \times_{Z_0} Z_{n_k}$$

- $\mathbb{B} = \mathbb{1}$, T = free monoid: Δ_T = (Lawvere theory)^{op}

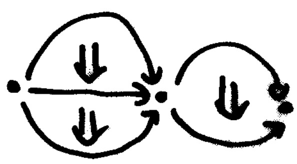
- $\mathbb{B} = \mathbf{1}$, T = free M-set: $\Delta_T = M^{op}$.

EXAMPLE: STRICT n-CATS

$\mathbb{B} = (\bullet \rightrightarrows \cdots \rightrightarrows \bullet)$, $\hat{\mathbb{B}} = n\text{-}\mathbf{Gph}$,

T = free strict n-cat

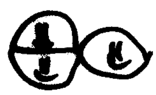
- Object of Δ_T : globular pasting diagram, e.g.



- Map in Δ_T : strict n-functor.

Have

$$N : \text{Str-}n\text{-Cat} \hookrightarrow \widehat{\Delta_T},$$

(NX) () = { labelled diagrams of shape  in X }

giving

$$\text{Str-}n\text{-Cat} \simeq \{ Z \in \widehat{\Delta_T} \mid Z \text{ preserves certain limits} \}.$$

In fact, $\Delta_T \simeq \Theta$ (Joyal, Berger, Makkai-Zawadowski).

DEFINITIONS OF WEAK n-CAT

Two styles:

① an n-cat is an algebra for a fam. rep. monad

② an n-cat is a presheaf with properties.

When $n=1$:

① a cat is a graph with structure

② a cat is a simplicial set with properties.

Idea in ②: presheaves are taken on some $\mathbb{D} \subseteq$ Str-n-Cat, and "the functor is

$$\mathbb{D}^{op} \longrightarrow \mathbf{Set}$$
$$S \longmapsto \{\text{weak functors } S \to X\}$$

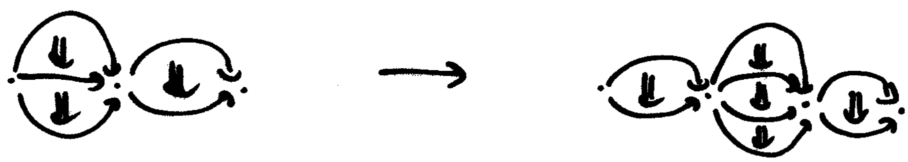
where X is our weak n-cat."

$① \hookrightarrow ②$

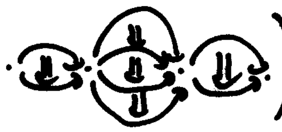
$\mathbb{B} = (\bullet \rightrightarrows \cdots \rightrightarrows \bullet)$, $\quad \hat{\mathbb{B}} = n\text{-}\mathrm{Gph}$,

T = free weak n-cat

- Object of Δ_T : globular pasting diagram
- A map

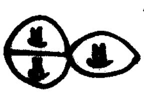


in Δ_T is a <u>strict</u> n-functor

F_{wk} () $\longrightarrow$ F_{wk} ().

Have

$$N : \text{Wk-}n\text{-Cat} \hookrightarrow \widehat{\Delta_T},$$

(NX) () = { strict n-functors F_{wk}() $\to X$ }

"= { weak n-functors F_{str}() $\to X$ }".