PhD topics

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This document is intended for students considering doing a PhD with me.

General A PhD is an apprenticeship in research, and I view it as an intellectual adventure. I will provide as much guidance as I can, but the ideal is that the choice of topic(s) is yours, and you eventually become the world expert on whatever it is you choose.

It is not necessary to choose a topic before beginning a PhD. One feature of mathematics is that it is very hard to know what a particular topic is or feels like, or whether you like working on it, until you actually begin. So usually, students doing a PhD with me would begin by dipping into a succession of topics, until they find one that particularly takes their interest and looks suitable for writing a thesis on. But every individual is different, and I'm always willing to be flexible.

The topics below Below I have listed some potential areas of investigation. I want to emphasize that these are not intended to be neat 3.5-year projects. Many are quite speculative. They may be too easy, or too hard, or ill-defined, or blind alleys. It's near-impossible to tell in advance how they will turn out. That's the nature of research!

I have not given references, but for most of the subjects listed, you can find relevant blog posts, talk slides or papers via my web page.

Practicalities For this, see the School website. Information is currently at http://www.maths.ed.ac.uk/school-of-mathematics/studying-here/pgr. Some quick points: (1) Application is done centrally, that is, through the School of Mathematics rather than me. I have no personal PhD places. (2) If you apply, you may have to choose a research group. Either 'Algebra and Number Theory' or 'Geometry and Topology' works. Which you choose makes little difference.

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1 Category theory

Automatic definitions Loosely, a group is a set equipped with whatever algebraic structure $\mathrm{Aut}(X)$ is equipped with, for an arbitrary set X, subject to whatever equations the structure on $\mathrm{Aut}(X)$ satisfies. In many introductory group theory classes, this is (essentially) how the definition of group is motivated: by abstracting the features of $\mathrm{Aut}(X)$. But can it be made precise? Is there a way of turning the first sentence into an exact statement?

The general question has nothing to do with groups; that is just one example. Similarly, one might seek to 'define' a ring by saying that the set of endomorphisms of an arbitrary abelian group is a typical ring, or 'define' a distributive lattice by saying that the set of linear subspaces of a vector space is a typical distributive lattice. So, the idea is that one mentions a typical such-and-such, puts this into a machine, and the machine spits out the general definition of a such-and-such. The challenge is to build that machine.

This may have something to do with codensity monads.

Duality and the Pythagorean tensor In the early 1970s, Lawvere made the provocative observation that a metric space can be regarded as an enriched category of a certain type. Now, there are many things that one might mean by the 'product' of two metric spaces A and B, and in particular, the cartesian product of sets $A \times B$ carries at least three interesting metrics:

$$\begin{aligned} d_1((a,b),(a',b')) &= d(a,a') + d(b,b'), \\ d_2((a,b),(a',b')) &= \sqrt{d(a,a')^2 + d(b,b')^2}, \\ d_\infty((a,b),(a',b')) &= \max\{d(a,a'), d(b,b')\}. \end{aligned}$$

The first and third metrics arise naturally from general categorical constructions when one views metric spaces as enriched categories. But what about the second? Arguably this is the most important, as it gives rise to the usual Euclidean metric on $\mathbb{R}^n$. (Lawvere calls it the 'Pythagorean tensor'.) What is the categorical explanation?

Eventual image Take an object X of a category and an endomorphism T of X. Consider the diagram

$$\cdots \xrightarrow{T} X \xrightarrow{T} X \xrightarrow{T} \cdots .$$

When X is suitably finite, the limit and colimit of this diagram coincide. How are we to understand this? What can be said without finiteness? These questions can be viewed from a categorical viewpoint or through the lens of dynamical systems.

2 Entropy and diversity

The categorical origins of entropy Ever since Shannon entropy was introduced in the 1940s, many theorems have been proved characterizing it uniquely. It can also be shown that Shannon entropy is inevitable from a categorical point of view, using the language of operads. But Shannon entropy is not the only important type of entropy. We might also look for the categorical/operadic origins of Rényi entropies, or of relative entropy, for instance.

Internal algebras More specifically, Shannon entropy and its scalar multiples can be characterized as the internal algebras in a certain categorical Δ-algebra, where Δ is the operad of simplices. This suggests taking other important operads P and examining the internal algebras in categorical P-algebras.

Unique characterizations of entropies and diversities Many different ways of measuring biological diversity have been proposed, often without an enormous amount of mathematical investigation. Often, these diversity measures are closely related to entropies of various kinds. The ideal is to be able to say 'if you want your diversity measure to satisfy properties X, Y and Z, then it must be this one'.

Since modern diversity measures often go beyond classical kinds of entropy, this is a new mathematical challenge. It is not even clear what techniques to bring to bear. Much of the related classical work is in the field of functional equations. But some recent innovations have been made by using probabilistic methods (specifically, large deviation theory), and it may well be that these methods can be applied here.

Diversity measures incorporating 'value' Initial efforts towards characterizing diversity measures suggest that the natural mathematical setting involves some notion of the 'value' of the units concerned (species, in the biological setting). When measuring ecological diversity, it is important to factor in the similarity between species; in that context, a species is more 'valuable' when it is both rare and dissimilar from most other species. But it is easy to think of other biological intepretations of 'value'. So, this greater level of generality appears to be biologically desirable as well as mathematically natural.

3 Magnitude of metric spaces

Magnitude is a numerical invariant of enriched categories. It makes sense for many kinds of enriched category, and is especially interesting for metric spaces. It can be viewed as a general notion of size, encompassing fundamental invariants such as cardinality and Euler characteristic.

The 2-dimensional convex magnitude conjecture Numerical experiments suggest that the magnitude of a compact convex subset of $\mathbb{R}^2$ is a certain linear combination of its area, its perimeter and its Euler characteristic. Is this true?

(Initially a more general conjecture had been made, involving convex subsets of $\mathbb{R}^n$ for general n. It turned out to be false in dimensions ≥ 5, but the question is still open in dimensions ≤ 4.)

The ℓ_1 convex magnitude conjecture The conjectures above all use the Euclidean metric on $\mathbb{R}^n$. But one can instead give $\mathbb{R}^n$ the ℓ_1 (taxicab/Manhattan) metric. A version of the convex magnitude conjecture *is* then true for arbitrary n, but only under certain technical restrictions that it would be nice to remove. Is it true in general?

Magnitude and geometric measure in $\mathbb{R}^n$ Recent work by Barceló and Carbery showed that for a compact subset A of $\mathbb{R}^n$, the volume of A can be recovered from the function $t \mapsto |tA|$, where tA means A scaled by a factor of $t > 0$ and $|tA|$ means the magnitude of tA. In brief: volume can be recovered from magnitude. More recent work still, by Gimperlein and Goffeng, showed that under suitable hypotheses, surface area can also be recovered from magnitude.

But there are other important geometric measures, besides volume and surface area. Can they, too, be recovered from magnitude?

All the work in this direction so far is heavily analytical, outside my expertise. However, Carbery is at Edinburgh and Gimperlein is at our sister department at Heriot-Watt University (also in the city of Edinburgh), so a collaborative supervision arrangement may be possible.

Dimension at different scales Given a compact set A in $\mathbb{R}^n$, the function $t \mapsto |tA|$ grows as fast as $t^{\dim A}$, where $\dim A \in \mathbb{R}$ is the Minkowski dimension of A and we are looking here at the asymptotic growth as $t \to \infty$. (This is a theorem of Meckes.) However, one can also ask how fast $|tA|$ is growing at a *particular* value of t.

For instance, let A consist of twelve points spaced equally in a circular pattern, as on a clock face. When A is shrunk very small, it looks like a single point (0-dimensional), and when A is blown up very large, all one sees is the individual points (so again, 0-dimensional). But at medium scales, it looks something like a circle (so, roughly 1-dimensional). In this particular example, the magnitude function $t \mapsto |tA|$ does indeed show this behaviour. And there are other examples strongly suggesting that magnitude gives a good notion of 'dimension at different scales'. But so far, there are almost no theorems on this topic at all.

Persistent homology and topology of data There is a fast-expanding field of algebraic topology applied to data, where one of the major aims is to detect topological features of discrete data-sets. (For instance, we should be able to detect that the 12-point space above 'looks like' a circle.) Persistent homology is an important tool in this field. There are many apparent points of commonality between the study of metric spaces using magnitude and the study of metric spaces using persistent homology. But this has never been investigated in more than the most superficial way.

Maximum diversity as a metric invariant Magnitude is closely related to diversity measures. More specifically, given a finite metric space, it is a theorem that the maximum diversity among all probability distributions on that space is equal to the magnitude of a certain subspace. But this is only for finite spaces. It remains to be seen how and whether the theorem extends to more general spaces.

Many of the topics above are about uncovering the geometric meaning of the magnitude of a metric space. Once maximum diversity has been extended from finite to more general metric spaces, one can similarly ask for the geometric meaning of maximum diversity—and whether it, like magnitude, encodes many existing geometric invariants.

Asymptotic inclusion-exclusion Another outstanding conjecture in the field of magnitude is whether

$$|t(A \cup B)| + |t(A \cap B)| - |tA| - |tB| \to 0$$

as $t \to \infty$, for suitable subsets $A, B \subseteq \mathbb{R}^n$. This would be natural for an 'invariant of size', and is consistent with all the results on magnitude of subsets of $\mathbb{R}^n$ that are known so far.

4 Other aspects of magnitude

Other enriched categories Besides metric spaces, there are other types of enriched category for which magnitude is worth investigating. This has already been done (to a greater or lesser extent) for sets, posets, graphs, groupoids, ordinary categories, and linear categories. Doubtless there is more to be said on some of these. There are also other types of enriched category for which magnitude has hardly been investigated at all, such as topological categories, n-categories, and categories enriched over posets.

Magnitude of graphs In particular, the general definition of magnitude provides a graph invariant. It assigns to each (finite, simple) graph a rational function over $\mathbb{Q}$. This appears to encode rather different properties of a graph than classical graph invariants such as the Tutte polynomial.

Some nontrivial results on graph magnitude are known, and some of those have already been revealed to be shadows of more sophisticated results on 'magnitude homology' of graphs (see below). But there are huge gaps in our understanding. This topic would suit someone with existing knowledge of graph theory.

Other connections The magnitude of an invertible square matrix Z is equal to the sum of all the entries of the inverse matrix Z^{-1}. This formula is fundamental to everything about magnitude. It is curious that the same construction (invert a matrix then sum its entries) appears in several other areas of mathematics. In those cases, it is natural to ask whether there is some connection with magnitude. Affirmative answers to this have already been given in a couple of cases (Cartan matrices in the representation theory of associative algebras, and similarity matrices in the measurement of biological diversity).

One outstanding case is that of effective sample size in statistics. In a certain context, the maximum possible effective sample size of an unbiased estimator is obtained by inverting a correlation matrix and summing its entries. Whether this can be brought into the enriched-categorical framework of magnitude remains unknown.

5 Magnitude homology

Foundations Shulman recently proposed (on the n-Category Café) a theory of magnitude (co)homology, in the wide generality of enriched categories. Magnitude homology decategorifies to give magnitude; or put another way, magnitude is the Euler characteristic of this homology theory.

This work is very recent (September 2016) and has not yet been published. It is potentially an extremely significant development. Fundamental aspects of it still need to be sorted out, including details of the decategorification step.

Magnitude homology of metric spaces Another pressing question is how to interpret the magnitude homology of metric spaces. So far, there is a result to the effect that for subsets $X \subseteq \mathbb{R}^n$, the first homology vanishes if and only if X is convex. The second magnitude homology of a metric space appears to be related to *how many* geodesic paths there are between pairs of points (not just whether they exist). But far more is unknown than known.

Magnitude homology of other enriched categories Before the general theory was proposed, Hepworth and Willerton worked out a theory of magnitude homology of graphs (seen as enriched categories). This should now be revisited in the light of general magnitude homology. And most certainly, magnitude homology should be investigated for other types of enriched categories, such as linear categories (especially those arising in representation theory).

The Euler characteristic of a category There is also a notion of the Euler characteristic of a (suitably finite) category. Again, this deserves revisiting in the light of the new general theory of magnitude homology, since it fits naturally into that larger framework.

6 Category theory and functional analysis

Lebesgue integration There is a clean universal characterization of the space L^1 of Lebesgue-integrable functions, bypassing all the traditional preliminary steps in the definition of Lebesgue integrability. This universal property immediately gives us Lebesgue integra*tion*, too. But it remains to be seen just how much else can be derived from the universal property. For instance, convolution is a very important operation; can we obtain this too?

Square-integrable functions on a topological group Given a locally compact abelian group G, one often wants to consider the space $L^2(G)$ of square-integrable functions on G, where the integration is taken with respect to Haar measure on G. Does the construction $G \mapsto L^2(G)$ have a simple universal property?

Other universal properties The animating principle here is that if a construction is found important by mathematicians in a particular field, then it is quite likely that it has a categorical explanation. ('What is socially important is categorically natural.') This is not *necessarily* the case, because sometimes constructions are particular to a certain subject and simply do not generalize beyond it. But it is often the case, even in areas such as analysis where historically, the presence of category theory has been rather slight.

I am fond of theorems of this type, stating universal properties of objects that are known to be mathematically important. It is hard to simply go looking for them, but if one were to do so, then (functional) analysis is probably a good area to try.

7 Integral geometry

Integral geometry involves the study of invariants such as perimeter, surface area, and their higher-dimensional generalizations (such as mean width). The word 'integral' refers to integration, not integers. It is related to geometric measure theory. In the most well-explored setting, one explores continuous functions ('valuations')

$$\mu\colon \{\text{convex subsets of } \mathbb{R}^n\} \to \mathbb{R}$$

that satisfy the inclusion-exclusion principle and are isometry-invariant. A theorem of Hadwiger classifies all such valuations.

Integral geometry in the 1-norm When investigating the magnitude of subsets of $\mathbb{R}^n$ with the 1-norm, I needed to take some of the classical integral geometry of Euclidean space (that is, $\mathbb{R}^n$ with the 2-norm) and create a parallel theory for the 1-norm. There are very striking parallels between integral geometry in the 2- and 1-norms, and they are *not* shared by the p-norm for any real value of $p \neq 1, 2$. For instance, the results in the 1- and 2-norms are very like each other but very unlike results in the 1.5-norm.

A puzzling aspect is that even though the *results* in the 1- and 2-norms are closely analogous, the *proofs* are substantially different. They each use particular geometric properties of the norm concerned. So, why do the results turn out to be so similar? Is there a common generalization?

Integral geometry in the ∞-norm Although the 1- and 2-norms behave very differently from the p-norm for other *finite* values of p, there are certain hints suggesting that $p = \infty$ behaves similarly to $p = 1$ and $p = 2$. At present there is no theory of integral geometry in the ∞-norm at all! So, the task is to create one and discover how it compares.

The algebra of translation-invariant valuations In the last decade or two, it has been discovered that the space of smooth, translation-invariant valuations on $\mathbb{R}^n$ carries a remarkable amount of algebraic structure: two types of multiplication, two types of duality, and (if we restrict to valuations invariant under a larger group of isometries) two types of comultiplication. All of this structure hangs together in an interesting way.

These results (largely due to Alesker and Fu) are very striking but also rather technical. It is tempting to ask for a more abstract explanation of where all this structure comes from, standing back from the technical details. Such an explanation may have to be preceded by some universal (or other categorical) characterizations of the structures involved.

The algebra of translation-invariant valuations in the 1-norm The work just described applies to $\mathbb{R}^n$ with its usual, Euclidean norm. But it turns out that much of it has an analogue in $\mathbb{R}^n$ with the 1-norm. In that setting, everything simplifies dramatically and the technicalities disappear. For instance, smoothness is no longer a consideration, and the vector space of translation-invariant valuations becomes finite-dimensional.

However, part of the story is about duality, and the linear dual of $\mathbb{R}^n$ with the 1-norm is $\mathbb{R}^n$ with the ∞-norm. For this reason, it is probably not possible to finish this story until the integral geometry of $\mathbb{R}^n$ in the ∞-norm has been developed.

If all this can be achieved, the payoff will be a simplified setting in which many of these technically forbidding results have easy-to-understand analogues, and where the theory exhibits the same formal structure as in the harder Euclidean setting.