Codensity and the ultrafilter monad

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These slides: available on my web page

$n$-Category Café posts: 1, 2, 3, 4

The moral of this talk

Whenever you meet a functor,
ask
“What is its codensity monad?”
Plan

1. What codensity monads are

With codensity monads as part of our toolkit:

   The notion of . . . automatically gives rise to the notion of . . .

2. finiteness of a set
3. finite-dimensionality of a vector space
4. finiteness of a family

   ultrafilter
double dualization
ultraproduct
1. What codensity monads are

(Isbell, Ulmer; Appelgate & Tierney, A. Kock)
Loosely

The codensity monad of a functor $G : \mathcal{B} \to \mathcal{A}$ is what the composite of $G$ with its left adjoint would be if $G$ had a left adjoint.

**Grammar:** given a functor $G : \mathcal{B} \to \mathcal{A}$, the codensity monad $T^G$ of $G$ is a certain monad on $\mathcal{A}$.

It is defined as long as $\mathcal{A}$ has enough limits.

The definition will be given later.
Characterization of the codensity monad

Motivation: Let $G : \mathcal{B} \to \mathcal{A}$ be a functor that does have a left adjoint, $F$. We have categories and functors

![Diagram]

and this is initial among all maps in $\text{CAT} / \mathcal{A}$ from $G$ to a monadic functor.

Theorem (Dubuc) Let $G : \mathcal{B} \to \mathcal{A}$ be a functor whose codensity monad $T^G$ is defined. Then

![Diagram]

is initial among all maps in $\text{CAT} / \mathcal{A}$ from $G$ to a monadic functor.

Corollary Let $G$ be a functor with a left adjoint, $F$. Then $T^G = G \circ F$. 
Three definitions of the codensity monad

Let $G : \mathcal{B} \to \mathcal{A}$ be a functor. Three equivalent definitions:

- The **codensity monad** of $G$ is the right Kan extension of $G$ along itself:

  $\mathcal{B} \xrightarrow{G} \mathcal{A} \xrightarrow{\downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow} \mathcal{A}$

  (and is defined iff the Kan extension exists).

- $T^G(A) = \int_B [\mathcal{A}(A, G(B)), G(B)] = \lim_{B \in \mathcal{B}, f : A \to G(B)} G(B)$.

- Recall: if $F : \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ small and $\mathcal{B}$ cocomplete, get adjunction

  $\mathcal{B} \xleftarrow{T} [\mathcal{A}^{\text{op}}, \text{Set}] \xrightarrow{\text{Hom}(F, -)} \xrightarrow{- \otimes F} \mathcal{A}^{\text{op}}, \text{Set}]$, e.g.

  $\text{Top} \xleftarrow{T} [\Delta^{\text{op}}, \text{Set}] \xrightarrow{|-|}$.
Three definitions of the codensity monad

Let $G: \mathcal{B} \rightarrow \mathcal{A}$ be a functor. Three equivalent definitions:

- The **codensity monad** of $G$ is the right Kan extension of $G$ along itself:

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{G} & \mathcal{A} \\
\downarrow & \searrow \searrow & \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\
\mathcal{A} & \xrightarrow{T_G} & \mathcal{A}
\end{array}
$$

(and is defined iff the Kan extension exists).

- $T_G^G(A) = \int_B \left[ \mathcal{A}(A, G(B)), G(B) \right] = \lim_{B \in \mathcal{B}, f: A \rightarrow G(B)} \mathcal{A}(A, G(B))$.

- If $\mathcal{B}$ is small and $\mathcal{A}$ is complete, get adjunction

$$
\begin{array}{ccc}
\mathcal{A} & \xleftarrow{\text{Hom}(\_, G)} & \mathcal{B}, \text{Set}^{\text{op}} \\
\downarrow & \searrow \uparrow & \\
\text{Hom}(\_, G) & \xrightarrow{\text{Hom}(\_, G)}
\end{array}
$$

and $T_G^G$ is the induced monad on $\mathcal{A}$. 
Two short but nontrivial examples

1. Let $\mathcal{A}$ be a category and $X \in \mathcal{A}$. The codensity monad of $1 \xrightarrow{X} \mathcal{A}$ is the endomorphism monad $\text{End}(X)$ of $X$, given by

$$\left(\text{End}(X)\right)(A) = [\mathcal{A}(A, X), X].$$

By Dubuc, for any monad $S$ on $\mathcal{A}$, an $S$-algebra structure on $X$ amounts to a map of monads $S \xrightarrow{} \text{End}(X)$.

2. The codensity monad of $G : \text{Field} \hookrightarrow \text{CRing}$ is given by

$$T^G(A) = \prod_{p \in \text{Spec}(A)} \text{Frac}(A/p)$$

($A \in \text{CRing}$). For example,

$$T^G(\mathbb{Z}) = \mathbb{Q} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) \times \cdots,$$

$$T^G(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/\text{rad}(n)\mathbb{Z}$$

where $\text{rad}(n)$ is the product of the distinct prime factors of $n$. 
2. Ultrafilters
What ultrafilters are

Lemma (Galvin and Horn) Let $X$ be a set and $\mathcal{U} \subseteq \mathcal{P}(X)$. The following are equivalent:

- $\mathcal{U}$ is an ultrafilter
- whenever $X = X_1 \amalg \cdots \amalg X_n$, there is a unique $i$ such that $X_i \in \mathcal{U}$.

There is a monad $U$ on $\text{Set}$, the ultrafilter monad, with

$$U(X) = \{\text{ultrafilters on } X\}$$

($X \in \text{Set}$).
Ultrafilters as measures

Let \( X \in \text{Set} \) and \( \mathcal{U} \in U(X) \). Think of elements of \( \mathcal{U} \) as ‘sets of measure 1’.

**Lemma (everyone)** An ultrafilter on a set \( X \) is essentially the same thing as a finitely additive probability measure on \( X \) taking values in \( \{0, 1\} \).

If an ultrafilter is a kind of measure, what is integration?

Given a finite set \( B \), define

\[
\int_X f \, d\mathcal{U} : \text{Set}(X, B) \longrightarrow B
\]

as follows:

for \( f \in \text{Set}(X, B) \), let \( \int_X f \, d\mathcal{U} \) be the unique element of \( B \) whose fibre under \( f \) belongs to \( \mathcal{U} \).

**Justification of terminology:** This ‘integration’ is uniquely characterized by:

- the integral of a constant function is that constant; and
- changing a function on a set of measure 0 doesn’t change its integral.
Measures correspond to integration operators

Let $X$ be a set. Given $\mathcal{U} \in U(X)$, we obtain a family of maps

$$\left( \text{Set}(X, B) \xrightarrow{\int_X - d\mathcal{U}} B \right)_{B \in \text{FinSet}}$$

natural in $B$. That is: given $\mathcal{U} \in U(X)$, we obtain an element

$$\int_X - d\mathcal{U} \in T^G(X)$$

where $T^G$ is the codensity monad of $G : \text{FinSet} \hookrightarrow \text{Set}$. So, we have

$$U(X) \quad \xrightarrow{\mathcal{U}} \quad T^G(X)$$

In fact, this defines an isomorphism of monads $U \xrightarrow{} T^G$. Hence:

**Theorem** (i) (Kennison and Gildenhuys) The codensity monad of $\text{FinSet} \hookrightarrow \text{Set}$ is the ultrafilter monad.

(ii) (Manes) The algebras for this monad are the compact Hausdorff spaces.
Moral of this section

The notion of finiteness of a set automatically gives rise to the notions of ultrafilter and compact Hausdorff space.
3. Double dualization
The linear analogue of the ultrafilter theorem

**Theorem** (i) The codensity monad of $\text{FDVect} \hookrightarrow \text{Vect}$ is double dualization. (ii) The algebras for this monad are the linearly compact vector spaces (certain topological vector spaces).

Table of analogues:

<table>
<thead>
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<td>compact Hausdorff spaces</td>
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Moral of this section

The notion of finite-dimensionality of a vector space automatically gives rise to the notions of double dualization and linearly compact vector space.
4. Ultra-products
What ultraproducts are

Let \( S = (S_x)_{x \in X} \) be a family of sets.

An element of the product \( \prod S = \prod_{x \in X} S_x \) is a family of elements \( (s_x)_{x \in X} \).

Now let \( \mathcal{U} \) be an ultrafilter on \( X \). We’ll define the ultraproduct \( \prod_{\mathcal{U}} S \).

Informally: An element of \( \prod_{\mathcal{U}} S \) is a family of elements \( (s_x) \) defined almost everywhere and taken up to almost everywhere equality.

Formally: An element of \( \prod_{\mathcal{U}} S \) is an equivalence class of families \( (s_x)_{x \in Y} \) with \( Y \in \mathcal{U} \), where

\[
(s_x)_{x \in Y} \sim (t_x)_{x \in Z} \iff \{ x \in Y \cap Z : s_x = t_x \} \in \mathcal{U}.
\]

Alternatively: \( \prod_{\mathcal{U}} S \) is the colimit of

\[
(U, \subseteq)^{op} \to \text{Set} \quad Y \mapsto \prod_{x \in Y} S_x.
\]

Can define ultraproducts similarly in any category with enough (co)limits.
The ultraproduct monad

Let $\mathcal{E}$ be a category with small products and filtered colimits.

Define a category $\text{Fam}(\mathcal{E})$ as follows:

- an object is a family $(S_x)_{x \in X}$ of objects of $\mathcal{E}$, indexed over some set $X$
- a map $(S_x)_{x \in X} \to (R_y)_{y \in Y}$ is a map of sets $f : X \to Y$ together with a map $R_{f(x)} \to S_x$ for each $x \in X$.

Given a family $S = (S_x)_{x \in X}$ of objects of $\mathcal{E}$, we get a new family

$$\left( \prod_{U \in \mathcal{U}(X)} S \right)_{U \in \mathcal{U}(X)}$$

of objects of $\mathcal{E}$.

Fact (Ellerman; Kennison) This assignation is part of a monad on $\text{Fam}(\mathcal{E})$, the ultraproduct monad for $\mathcal{E}$. 
Ultraproducts are inevitable

Let $\text{FinFam}(\mathcal{E})$ be the full subcategory of $\text{Fam}(\mathcal{E})$ consisting of the families $(S_x)_{x \in X}$ in which $X$ is finite.

**Theorem (i) (with Anon.)** The codensity monad of $\text{FinFam}(\mathcal{E}) \hookrightarrow \text{Fam}(\mathcal{E})$ is the ultraproduct monad.

(ii) (Kennison) When $\mathcal{E} = \textbf{Set}$, the algebras for this monad are the sheaves on compact Hausdorff spaces.
Moral of this section

The notion of finiteness of a family automatically gives rise to the notions of ultraproduct and sheaf on a compact Hausdorff space.
Summary
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- The codensity monad of a functor $G$ is a substitute for $G \circ F$ (where $F \dashv G$) that makes sense even when $G$ has no left adjoint.

- Routinely asking ‘what is the codensity monad?’ is worthwhile.

- For example, it establishes that the following concepts are categorically inevitable:
  
  - ultrafilter
  - double dual vector space
  - ultraproduct
  - compact Hausdorff space
  - linearly compact vector space
  - sheaf on a compact Hausdorff space.