Coalgebraic Topology

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Cats 3, Pisa

Plan

1. Some basic objects, and their universal properties

2. Recursively-defined spaces

   - sketch of a large theory

3. Speculation

1. Some basic objects, and their universal properties

In homotopy theory:

- Given a space S, have notion of *path* in S.
- Every path has a *starting* and *finishing* point.
- Can *compose* two adjoining paths.

1. Some basic objects, and their universal properties

In homotopy theory:

- Given a space S, have notion of path in S. $[0,1] \to S$
- Every path has a starting and finishing point. $0, 1 \in [0,1]$
- Can compose two adjoining paths.

$$[0,1] \xrightarrow{\times 2} [0,2] = \frac{[0,1] \amalg [0,1]}{\text{first } 1 \sim \text{second } 0} = [0,1] \vee [0,1]$$

1. Some basic objects, and their universal properties

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- Every path has a starting and finishing point. $0, 1 \in [0,1]$
- Can compose two adjoining paths. $[0,1] \xrightarrow{\times 2} [0,2] = \dfrac{[0,1] \amalg [0,1]}{\text{first } 1 \sim \text{second } 0} = [0,1] \vee [0,1]$
- Endpoints $0, 1$ of $[0,1]$ are distinct.
- Endpoints $0, 1$ of $[0,1]$ are closed.

1. Some basic objects, and their universal properties

<u>In homotopy theory:</u>

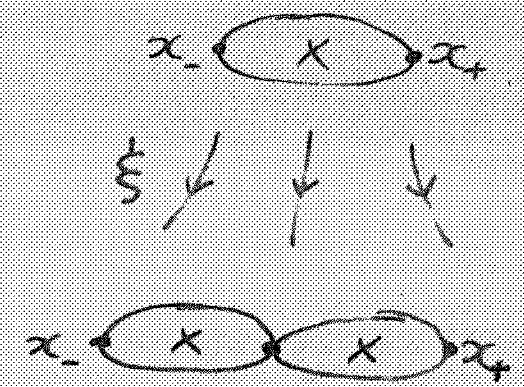
So $[0,1]$ comes equipped with distinct, closed basepoints 0 & 1,

and a basepoint-preserving map $[0,1] \xrightarrow{\gamma} [0,1] \vee [0,1]$.

Let $\mathcal{D}$ be the category in which an object is

a space X equipped with distinct, closed basepoints x_- & x_+,

and a basepoint-preserving map $X \xrightarrow{\xi} X \vee X$.



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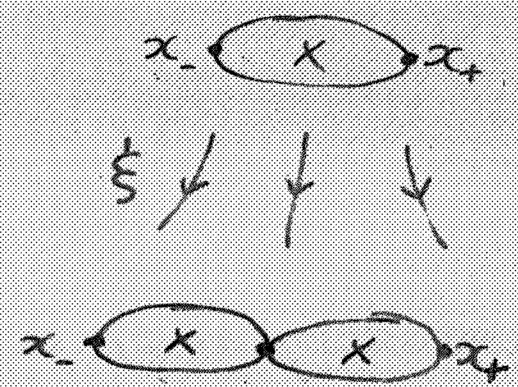

Thm (Freyd + ε):

The terminal object of $\mathcal{D}$ is $[0,1]$

(with the structure described above)

1. Some basic objects, and their universal properties

Digression: terminology

Let $\mathcal{C}$ be a category and $G: \mathcal{C} \longrightarrow \mathcal{C}$ an endofunctor.

A G-coalgebra is a pair $(X \in \mathcal{C}, \ \xi: X \longrightarrow G(X))$.

(Maps of G-coalgebras are defined in obvious way.)

E.g.: $\mathcal{C} =$ (spaces equipped with 2 distinct, closed basepoints), $G(X) = X \vee X$. Then $\mathcal{D} = \mathrm{Coalg}(G)$, and $[0,1]$ is the terminal coalgebra.

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the terminal coalgebra.

<u>Lemma</u> (Lambek): If (U, γ) is a terminal G-coalgebra then
$\gamma : U \to G(U)$ is an isomorphism.

<u>E.g.</u>: Our map $[0,1] \longrightarrow [0,2] = [0,1] \vee [0,1]$ is an isomorphism.

1. Some basic objects, and their universal properties

In homotopy theory:

Many spaces are built by gluing the topological simplices

$\Delta^0, \Delta^1, \ldots$ along faces.

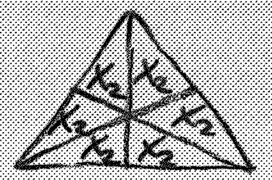
subcat of Δ: just face maps

The resulting functor $\Delta_{face} \xrightarrow{\Delta^\bullet} \mathrm{Top}$ is v. important!

$$[n] \longmapsto \Delta^n$$

Let $\mathcal{C}$ be the category of "nondegenerate" functors $\Delta_{face} \to \mathrm{Top}$.

Let $G : \mathcal{C} \to \mathcal{C}$ be "barycentric gluing":

$G(X)_0 = X_0$, $\quad G(X)_1 = X_1 \vee X_1$, $\quad G(X)_2 =$ 

$, \ldots$

Then $(\Delta^\bullet, \; \Delta^\bullet \xrightarrow{\cong} G(\Delta^\bullet))$ is the terminal G-coalgebra.

1. Some basic objects, and their universal properties

<u>In higher category theory:</u> (Simpson)

Given a strict ω-category X, we get a category enriched in <u>StrωCat</u>. (Its objects are the 0-cells of X.)

Hence there is a canonical functor

$$\underline{\mathbf{Str}\omega\mathbf{Cat}} \longrightarrow \underline{\mathbf{Str}\omega\mathbf{Cat}}\text{-}\mathbf{Cat}.$$

There is an endofunctor

$$\text{(finite product categories)} \circlearrowleft \quad \mathcal{V} \mapsto \mathcal{V}\text{-}\mathbf{Cat}$$

and its terminal coalgebra is <u>StrωCat</u>.

1. Some basic objects, and their universal properties

In functional analysis:

Let Ban_* be the category of pairs (real Banach space X, point $x \in X$ with $\|x\| \leq 1$), with the maps of norm ≤ 1.

Define $G : \mathrm{Ban}_* \circlearrowleft$ by $G(X, x) = (X \oplus X, (x, x))$, where the norm on $X \oplus X$ is $\|(x,y)\| = \frac{1}{2}(\|x\| + \|y\|)$.

Then a G-algebra consists of:

- a Banach space X
- a point $x \in X$ with $\|x\| \leq 1$
- a map $\xi : X \oplus X \to X$ of norm ≤ 1 such that $\xi(x, x) = x$.

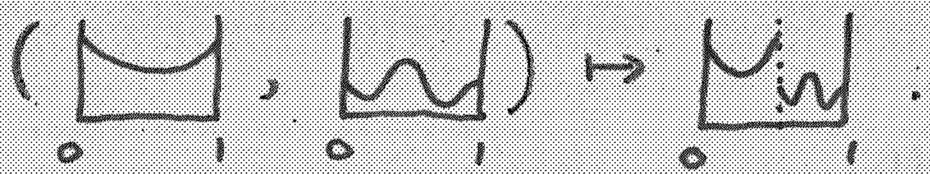
<u>E.g.</u>: $(\mathbb{R}, 1, \text{mean})$ is a G-algebra.

1. Some basic objects, and their universal properties

In functional analysis:

Thm: The initial G-algebra is $(L^1[0,1],\ 1,\ \gamma)$, where

$\gamma : L^1[0,1] \oplus L^1[0,1] \longrightarrow L^1[0,1]$ is "juxtapose and squeeze":



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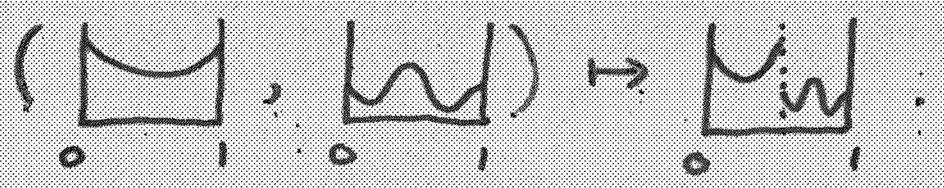

The unique G-algebra map

$$(L^1[0,1],\ 1,\ \gamma) \longrightarrow (\mathbb{R},\ 1,\ \text{mean})$$

is $\int_0^1$.

2. Recursively-defined spaces

Discrete case:

A <u>discrete equational system</u> is a pair (I, M) where I is a set and $M = (M_{ij}) \in \mathbb{N}^{I \times I}$ is a matrix such that $\forall i, \sum_j M_{ij} < \infty$.

Any such induces a functor

$$M \otimes - : \mathbf{Top}^I \longrightarrow \mathbf{Top}^I$$
$$X \longmapsto M \otimes X = \left(\coprod_j M_{ij} \times X_j \right)_{i \in I}$$

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<u>Idea:</u> "Solve the equations
$$X_i \cong \coprod_j M_{ij} \times X_j \quad (i \in I)",$$
i.e. look for the fixed points of $M \otimes -$.

<u>Better idea:</u> Look for the terminal $(M \otimes -)$-coalgebra.

An $(M \otimes -)$-coalgebra is called an <u>M-coalgebra</u>.

The terminal M-coalgebra is called the <u>universal solution</u>, $(\mathcal{U}, \gamma)$.

2. Recursively-defined spaces

Discrete case:

<u>E.g.:</u> $I = \{1\}$, $M = (2)$. Then the "system of equations" is

$$X_1 \cong 2 \times X_1 .$$

An M-coalgebra is a space X_1 plus a map $X_1 \longrightarrow X_1 \amalg X_1 = 2 \times X_1$.

The universal solution is the Cantor set:

$$U_1 = 2^{\mathbb{N}}, \qquad \gamma_1 : 2^{\mathbb{N}} \xrightarrow{\cong} 2 \times 2^{\mathbb{N}} .$$

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<u>Eg.:</u> $I = \{1, 2\}$, $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then the "system of equations" is

$$X_1 \cong X_1$$
$$X_2 \cong X_1 \amalg X_2 .$$

An M-coalgebra consists of spaces X_1, X_2 and maps $\left\{ \begin{matrix} X_1 \to X_1 \\ X_2 \to X_1 \amalg X_2 \end{matrix} \right\}$.

The universal solution (U, γ) has $\left\{ \begin{matrix} U_1 = \{\ast\}, \\ U_2 = \mathbb{N} \cup \{\infty\} = (\bullet \quad \bullet \ \bullet \!\bullet\!\bullet) \end{matrix} \right\}$.

2. Recursively-defined spaces

Discrete case:

E.g.: Consider walks on $\mathbb{N} = \{0, 1, \ldots\}$, of the following type:

- you start at some position n and walk forever
- every second, you take one step left or right
- but if you're at 0, you stay there.

Let W_n be the space of walks starting at n, with the profinite topology. Then $W_n \cong W_{n-1} \amalg W_{n+1}$ $(n \geq 1)$.

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There is an equational system with $I = \mathbb{N}$; informally, it's

$$X_0 \cong X_0$$
$$X_n \cong X_{n-1} \amalg X_{n+1} \quad (n \geq 1).$$

Its universal solution is W.

(In a certain precise sense, $(W_n)_{n=1}^{\infty}$ has period 6.)

2. Recursively-defined spaces

Discrete case:

DISCRETE EQUATIONAL SYSTEM

set $\mathbb{I}$

\+ family $(M_{ij})_{i,j \in \mathbb{I}}$ of nat nums

satisfying finiteness

Idea:

"$X_i \cong$ some coproduct of X_j's"

General case:

EQUATIONAL SYSTEM

small cat $\mathbb{I}$

\+ functor $M : \mathbb{I}^{op} \times \mathbb{I} \to \mathrm{FinSet}$

satisfying finiteness & nondegeneracy

Idea:

"$X(i) \cong$ some colimit of $X(j)$'s".

2. Recursively-defined spaces

Discrete case:

DISCRETE EQUATIONAL SYSTEM

set I

+ family $(M_{ij})_{i,j \in I}$ of nat nums

satisfying finiteness

<u>Idea:</u>

"$X_i \cong$ some <u>coproduct</u> of X_j's"

M-COALGEBRA

family $(X_i)_{i \in I}$ of spaces

+ map $\xi : X \longrightarrow M \otimes X$

$$\left(\xi_i : X_i \to \coprod_j M_{ij} \times X_j \right)$$

General case:

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M-COALGEBRA

nondegen functor $X : \mathbb{I} \longrightarrow \mathrm{Top}$

+ map $\xi : X \longrightarrow M \otimes X$

$$\left(\xi_i : X(i) \longrightarrow \int^j M(i,j) \times X(j) = \left(\coprod_j M(i,j) \times X(j) \right) / \sim \right)$$

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Set I

+ family $(M_{ij})_{i,j\in I}$ of nat nums

satisfying finiteness

Idea:

"$X_i \cong$ some coproduct of X_j's"

UNIVERSAL SOLUTION (TERMINAL M-COALG)

Always exists.

Constructed as a limit.

SPACES ARISING IN UNIV SOLNS

Totally disconnected compact metrizable spaces

General case:

EQUATIONAL SYSTEM

small cat $\mathbb{I}$

+ functor $M : \mathbb{I}^{op} \times \mathbb{I} \to \mathbf{FinSet}$

satisfying finiteness & nondegeneracy

Idea:

"$X(i) \cong$ some colimit of $X(j)$'s".

UNIVERSAL SOLUTION (TERMINAL M-COALG)

Sometimes exists (know when).

Construction less obvious.

SPACES ARISING IN UNIV SOLNS

Compact metrizable spaces

2. Recursively-defined spaces

<u>General case:</u>

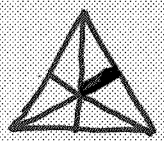
<u>E.g.:</u> Interval. Here $\mathbb{I} = (i \rightrightarrows j)$ and M "is" the system

$$X(i) \cong X(i),$$
$$X(j) \cong X(j) \amalg_{X(i)} X(j).$$

Universal solution (U, γ) has $U = (\{*\} \overset{0}{\underset{1}{\rightrightarrows}} [0,1])$.

<u>E.g.:</u> Topological simplices. Here $\mathbb{I} = \Delta_{face}$ and

$$M([p],[q]) = \text{"}\{p\text{-cells in the bary subdiv of } \Delta^q\}\text{"},$$

e.g.  $\in M([1],[2])$.

Universal solution (U, γ) has $U = \Delta^{\bullet} : \Delta_{face} \longrightarrow \mathbf{Top}$.

3. Speculation

Connectivity:

Let $(\mathbb{I}, M)$ be an equational system with universal solution (U, γ).

You can tell whether all the spaces $U(i)$ are connected by looking at the equations.

Indeed, the following are equivalent:

① all the spaces $U(i)$ are connected;

② all the "colimits on the RHS" are connected.

E.g.: Cantor set: binary coproduct is not a connected colimit, and the Cantor set is not connected.

Question: Can all the π_n's of the spaces $U(i)$ be read off from the equations?

3. Speculation

Euler characteristic:

Idea: Use the equations to calculate the Euler characteristics of the spaces $U(i)$.

E.g.: Interval: from $[0,1] \cong [0,1] \amalg_{\{*\}} [0,1]$

"deduce" $\chi([0,1]) = \chi([0,1]) - \chi(\{*\}) + \chi([0,1])$

i.e. $\chi([0,1]) = \chi(\{*\})$

$= 1$.

E.g.: Walks: have, similarly, $\chi(W_n) = \chi(W_{n-1}) + \chi(W_{n+1})$ $(n \geq 1)$,

$\chi(W_0) = 1$.

Deduce that $Re(\chi(W_n)) = Re(e^{n(i\pi/3)})$.

3. Speculation

Complex dynamics:

Every complex rational function f has a Julia set $J(f) \subseteq \mathbb{C}$.

Conjecture For every rational f,

(i) there are some finite equational system $(\mathbb{I}, M)$ and some $i \in \mathbb{I}$ such that $U(i) \cong J(f)$

(ii) $\chi(J(f))$ is well-defined $\in \mathbb{Q}$

(iii) $\chi(J(f)) \geq 0$.

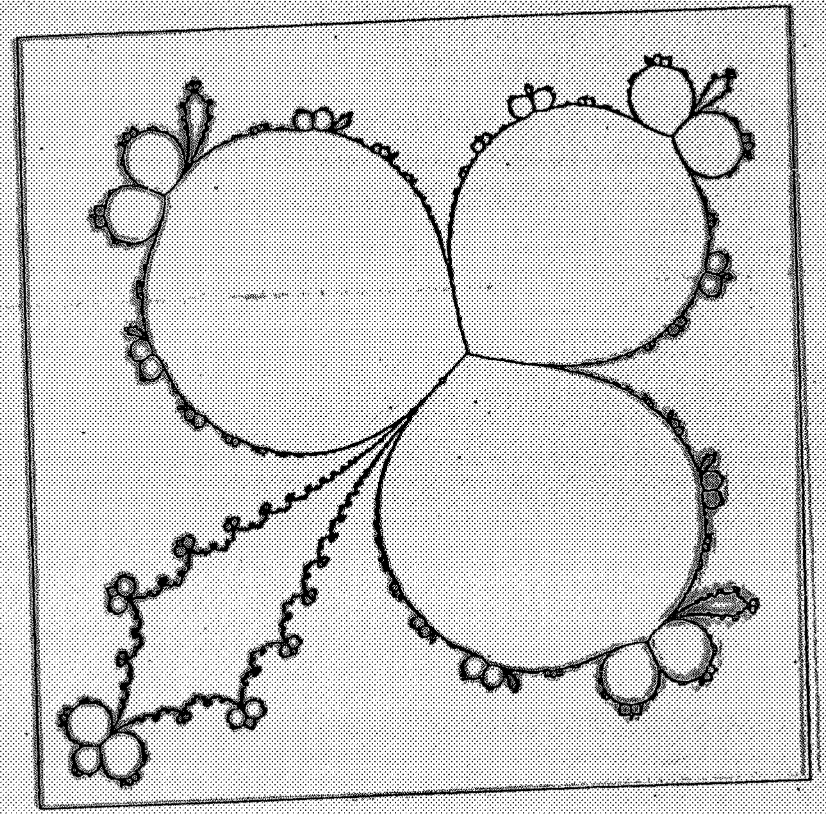


Image: Milnor

Julia set for $f(z) = z^5 + (.8 + .8i)z^4 + z$.