Review of *Coherence in Three-Dimensional Category Theory* by Nick Gurski (Cambridge University Press, 2013)*.

Nick Gurski's new book addresses some central concerns of the subject known as higher category theory; and yet, it is some distance from what many people now understand by that term. This may puzzle some. I will therefore begin by locating Gurski's book within the mathematical landscape.

Let $n \in \mathbb{N} \cup \{\infty\}$. Roughly, an n-category consists of some objects, some 1-morphisms between objects, some 2-morphisms between 1-morphisms (when those 1-morphisms have the same domain and codomain), and so on, up to n-morphisms between $(n-1)$-morphisms, or without end if $n = \infty$. These morphisms can be composed in various ways, and composition satisfies axioms. Thus, a 0-category is a set and a 1-category is a category.

The devil is in the detail. If we ask that the various compositions satisfy *strict* axioms, such as $h \circ (g \circ f) = (h \circ g) \circ f$, then we arrive at the definition of a so-called *strict* n-category. These are very well understood, but the definition excludes many natural examples. For instance, given a topological space X and $n \in \mathbb{N} \cup \{\infty\}$, we would like there to be an n-category $\Pi_n(X)$ in which the objects are the points of X, the 1-morphisms are paths, 2-morphisms are homotopies between paths, 3-morphisms are homotopies between homotopies, and so on. But concatenation of homotopies is not strictly associative or unital, so $\Pi_n(X)$ is not a strict n-category. We are therefore led to seek a definition of non-strict, or *weak*, n-category, that includes such examples.

This is where the landscape opens up. On one side, there are algebraic approaches to the problem. Here, an n-category is conceived as an algebraic structure, consisting of a collection of morphisms of each dimension, equipped with various operations satisfying universally quantified equations. On the other side, the non-algebraic approach does not attempt to assign a definite composite to each composable pair of morphisms, but merely asserts the existence of some third morphism satisfying a suitable universal property.

This distinction can be explained by analogy with cartesian products of sets. One person might take the approach that any two sets X and Y have a definite product $X \times Y$; but it should then be observed that the products $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are not actually equal, only canonically isomorphic, and that, moreover, these isomorphisms satisfy equations of their own (such as a pentagonal identity for four-fold products). A different person might assert that the product is only defined up to isomorphism; but then they need to state its characterizing universal property, and they lose the right to speak of a specific set called $X \times Y$, at least without further justification.

The algebraic approach goes back half a century, to Bénabou's definition of bicategory (weak 2-category) [1]. His work made plain one difficulty of this approach: in the definitions of bicategory, functor between bicategories, and so on, the coherence axioms (such as the aforementioned pentagon) are quite complicated. The complications multiply in dimension 3, as demonstrated by the

* *Bulletin of the London Mathematical Society*, to appear

1995 definition of tricategory (weak 3-category) by Gordon, Power and Street [2]. Their work, of which Gurski's is a successor, established a crucial fact: that while every bicategory is biequivalent to some strict 2-category, *not* every tricategory is triequivalent to some strict 3-category. The theory of weak 3-categories is, therefore, substantially deeper than the theory of strict 3-categories.

Once that had been discovered, the race was on to frame a definition of weak n-category for arbitrary n. Algebraic definitions were proposed by, among others, Penon, Batanin, and Leinster, all using techniques of categorical universal algebra to sidestep the ballooning complexity of the coherence diagrams. Non-algebraic definitions were also stated, by Simpson and Tamsamani (using simplicial sets), and by Baez and Dolan on the one hand and Hermida, Makkai, and Power on the other (using a different cell shape). References and a survey of the state of the art in 2001 can be found in [6]. All these proposed definitions are well-motivated, but the relationships between them are not fully understood.

However, many naturally occurring n-categories have the special property of being $(n, 1)$-categories, meaning that all m-morphisms for $m > 1$ are invertible (in a suitably weak sense). For instance, this is true of the ∞-category of topological spaces, continuous maps, homotopies, homotopies between homotopies, and so on. This suggests developing a free-standing theory of $(n, 1)$-categories, thus avoiding some of the difficulties associated with general n-categories. Joyal, building on Boardman and Vogt's work on weak Kan complexes, did just that [3], and his theory was further developed in the now well-known work of Lurie [7]. (Confusion has been caused by Lurie using '∞-category' to mean $(\infty, 1)$-category; thus, his '∞-categories' are a narrower concept than ∞-categories in general.)

A great strength of category theory is that it transcends the divides between different branches of mathematics. It is no more tied to topology, say, than it is to algebra or analysis or combinatorics. One early expression of the hoped-for unification that higher category theory would bring was Grothendieck's *homotopy hypothesis*. Roughly, this states that ∞-groupoids (that is, $(\infty, 0)$-categories) are essentially the same thing as topological spaces (of the kind that homotopy theorists like to consider), the correspondence being provided by Π_∞. It therefore asserts an equivalence between algebra (∞-groupoids) and topology (spaces).

The homotopy hypothesis can be made trivial by interpreting both '∞-groupoid' and 'space' to mean Kan complex. Both interpretations are reasonable in isolation. But for the homotopy hypothesis to have maximum substance, embodying 'algebra = topology' to the full, '∞-groupoid' must be interpreted algebraically and 'space' topologically (Figure 1).

Much recent work in higher category theory emphasizes an understanding of n-categories that is non-algebraic. As the case of the homotopy hypothesis illustrates, something is thereby missed. Similarly, for instance, it would be wasteful to discard the insight into n-categories provided by Penon's proposed definition, which pinpoints the position of the theory of n-categories within the world of all higher-dimensional algebraic theories. The conception of n-categories as algebraic structures risks not getting the attention it deserves.

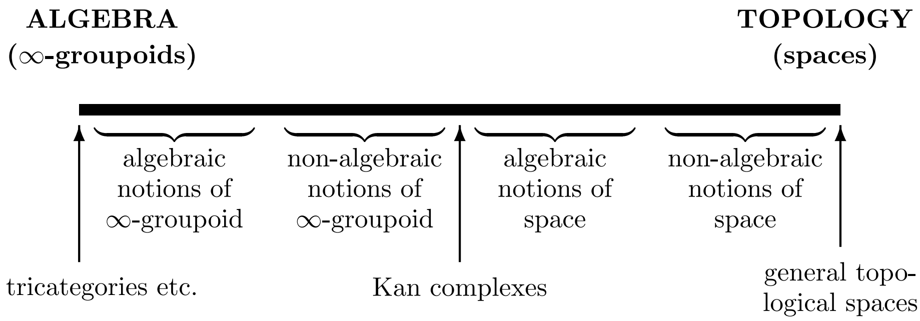

Figure 1: Algebraic and non-algebraic notions of ∞-groupoid and space.

Gurski's book is thoroughly algebraic. It is also a book-length endorsement of Gelfand's maxim: study the simplest nontrivial example. As the author says on page 1:

> From the perspective of a "hands-on" approach to defining weak n-categories, tricategories represent the most complicated kind of higher category that the community at large seems comfortable working with. On the other hand, dimension three is the lowest dimension in which strict n-categories are genuinely more restrictive than fully weak ones, so tricategories should be a sort of jumping off point for understanding general higher dimensional phenomena.

'Hands-on' is exactly what this book is: sophisticated abstract methods are used, but they are consistently given concrete expression in the three-dimensional setting. In fact, consistency of approach is one of the virtues of this work. For instance, Gordon, Power and Street's definition of tricategory falls just short of being fully algebraic; Gurski modifies it so that it is, which complicates the definition a little, but the resulting conceptual purity pays dividends later.

The book begins with a brisk forty-page review of the theory of bicategories. Assuming that the reader knows the basic definitions, it includes such topics as coherence for both bicategories and functors between them, and the Gray tensor product. It is no-nonsense, well-written, and covers a lot.

(Incidentally, the definition of orthogonal factorization system given as Definition 3.18 is much shorter than the usual one, but equivalent. It deserves to be better known. I believe it is due to Joyal [4].)

Part II, 'Tricategories', occupies nearly half the book. Its central theme is coherence (again building on the seminal work of Gordon, Power and Street). The word 'coherence' has two different but closely related meanings. The first is that 'all' diagrams commute. For example, the standard definition of monoidal category involves two axioms on the associativity and unit isomorphisms (one pentagonal and one triangular), and we know that no more are needed because it can be proved from just those two that *any* diagram built from these isomorphisms commutes.

The second meaning of coherence is that every weak structure is equivalent to some strict (or stricter) structure. For example, every monoidal category is equivalent to a strict monoidal category, and every bicategory is biequivalent to a strict 2-category. Although not every tricategory is triequivalent to a strict 3-category, a weaker coherence theorem holds: every tricategory is equivalent to a **Gray**-category (more on which below). An alternative coherence theorem has also been conjectured by Simpson and partially proved by Joyal and Kock [5]: every tricategory is triequivalent to one in which everything except the identities is strict.

Gurski takes pains to compare and contrast these two types of coherence. For example, the tricategory axioms are chosen in such a way that all sensible diagrams commute; that is why the axioms are what they are. But 'sensible' must be interpreted with care, and Gurski makes this point vividly with a specific example of a diagram that does not commute.

The rest of Part II consists largely of explanations of tricategorical facts, bridging the gap between the abstract and the concrete. Some of these facts are known, or at least folklore; almost everything is very clearly explained. The author is modest in not drawing attention to the substantial mathematical clarifications that he has contributed.

One of the highlights of Part II is the careful examination in Chapter 5 of three important tricategories: the tricategory of bicategories, the tricategory of topological spaces, and the fundamental 3-groupoid of a space. Another, occupying Chapters 9 and 10, is the sequence of coherence theorems for tricategories and for functors between them.

By the start of Part III, '**Gray**-monads', the book has moved definitively from creative exposition to new research. As noted above, **Gray**-categories can be viewed as semi-strict tricategories. But importantly, they are also categories enriched in the category **Gray** of 2-categories with the Gray tensor product, and this allows one to tap into the enormous power of enriched category theory.

In an especially lucid passage of the Introduction, Gurski describes the strengths and limitations of the enriched approach. One dimension down, he notes, a similar strategy has been exploited very effectively by the Sydney category theory school (especially Kelly and Lack) to study 2-categories and 2-monads on them. The author is explicit about which parts of that theory he has succeeded in reproducing in three dimensions, and which he has not.

Part III is technically sophisticated and, as Gurski frankly concedes, free of examples. An important role is played by codescent diagrams, which 'should be considered higher dimensional versions of coequalizers'. Just as every group has a canonical presentation (take all possible generators and relations), and more generally every algebra for a monad is canonically a coequalizer of free algebras, every lax algebra for a **Gray**-monad has a canonical codescent diagram. Following this train of thought leads to a canonical way of turning a lax algebra into a strict algebra—and eventually to a general coherence result for pseudo-algebras, the subject of the final chapter.

Throughout, the writing is disciplined, unfussy and direct, with no rambling. Many authors of research monographs cannot resist the temptation to digress, to loosen their belts a little. Gurski resists.

My greatest criticism concerns the index. At barely more than a page, there is not nearly enough of it, and there is no index of notation at all. A reader who wants to find out what the author means by 'functor of bicategories' or $\mathbf{Hom}(A, B)$, or know where the cited work of Joyal and Kock is discussed, will find no help here. Occasional slip-ups are inevitable, and poor indexing can exacerbate them. For instance, one might reach page 17 and read of 'the functor bicategory $\mathbf{Bicat}(B, C)$', thus far undefined. Ideally, one would turn to the index or index of notation and discover that it is defined on page 21, thus recovering gracefully from the error. But in reality, the index does not help.

Overall, though, this is a very well-written book, containing many significant new results and gems of exposition, as well as representing an important perspective on higher category theory.

References

[1] J. Bénabou, Introduction to bicategories. In *Reports of the Midwest Category Seminar*, Lecture Notes in Mathematics 47, Springer, Berlin, 1967.

[2] R. Gordon, A.J. Power and R. Street, Coherence for tricategories. *Memoirs of the American Mathematical Society* 117 (1995), no. 558.

[3] A. Joyal, Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra* 175 (2002), 207–222.

[4] A. Joyal, The theory of quasi-categories and its applications. In *Advanced Course on Simplicial Methods in Higher Categories*, Centre de Recerca Matemàtica, Barcelona, 2008.

[5] A. Joyal and J. Kock, Weak units and homotopy 3-types. In *Categories in Algebra, Geometry and Mathematical Physics*, Contemporary Mathematics 431, American Mathematical Society, Providence, RI, 2007.

[6] T. Leinster, A survey of definitions of n-category. *Theory and Applications of Categories* 10 (2002), 1–70.

[7] J. Lurie, *Higher Topos Theory.* Annals of Mathematics Studies 170, Princeton University Press, Princeton, NJ, 2009.

Tom Leinster, University of Edinburgh.