The Euler characteristic of an associative algebra

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Preview of the main theorem

Theorem

Let A be an algebra, of finite dimension and finite global dimension, over an algebraically closed field.

Then the magnitude of the linear category of projective indecomposable A-modules is equal to

$$\chi_A(S, S),$$

where

- χ_A is the Euler form of A;
- S is the direct sum of the simple A-modules (one per iso class).

But first, I will:

- explain why you might care;
- define the terms in red.

Why you might care

Many mathematical structures come with a canonical notion of size, e.g. cardinality of sets, dimension of vector spaces, volume of subsets of $\mathbb{R}^n$.

Schanuel, Rota and others made a convincing case that Euler characteristic belongs to this family.

> *Euler's analysis, which demonstrated that in counting suitably 'finite' spaces one can get well-defined negative integers, was a revolutionary advance in the idea of cardinal number—perhaps even more important than Cantor's extension to infinite sets, if we judge by the number of areas in mathematics where the impact is pervasive.*

—Stephen Schanuel

We might seek the canonical notion of size for associative algebras. Perhaps it will resemble other things that we call 'Euler characteristic'.

Plan

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2. Algebraic background
3. The theorem
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1. Categorical background

Monoidal categories

Informal definition A monoidal category $\mathbf{V}$ is a category equipped with an operation $\otimes$ of multiplication on the objects (and maps). It is required to be associative (in a reasonable sense) and have a unit object I.

Examples

- $\mathbf{V} = \mathbf{Set}$, $\otimes = \times$, $I = \{\star\}$. Or similarly with $\mathbf{V} = \mathbf{FinSet}$.
- $\mathbf{V} = \mathbf{Vect}_K$, $\otimes = \otimes_K$, $I = K$. Or similarly with $\mathbf{V} = \mathbf{FDVect}$.
- $\mathbf{V}$ is the category whose objects are the elements of $[0, \infty]$, with one map $x \longrightarrow y$ if $x \geq y$, and with no maps $x \longrightarrow y$ otherwise. This is monoidal with $\otimes = +$ and $I = 0$.

Enriched categories

Definition A category **A** consists of:

- a set/class ob **A** of objects
- for each $a, b \in \text{ob}\,\mathbf{A}$, a set $\text{Hom}(a, b)$
- for each $a, b, c \in \text{ob}\,\mathbf{A}$, a map

$$\text{Hom}(a, b) \times \text{Hom}(b, c) \longrightarrow \text{Hom}(a, c)$$

- for each $a \in \text{ob}\,\mathbf{A}$, a map

$$\{\star\} \longrightarrow \text{Hom}(a, a),$$

all subject to associativity and identity axioms.

Enriched categories

Fix a monoidal category $\mathbf{V}$.

Definition A $\mathbf{V}$-category $\mathbf{A}$ consists of:

- a set/class $\text{ob}\,\mathbf{A}$ of objects
- for each $a, b \in \text{ob}\,\mathbf{A}$, an object $\text{Hom}(a, b)$ of $\mathbf{V}$
- for each $a, b, c \in \text{ob}\,\mathbf{A}$, a map

$$\text{Hom}(a, b) \otimes \text{Hom}(b, c) \longrightarrow \text{Hom}(a, c)$$

- for each $a \in \text{ob}\,\mathbf{A}$, a map

$$I \longrightarrow \text{Hom}(a, a),$$

all subject to associativity and identity axioms.

Examples

- A **Set**-category is an ordinary category.
- A **Vect**-category is a linear category: hom-sets are vector spaces and multiplication is bilinear.
- Any metric space can be viewed as a $[0, \infty]$-category.

The magnitude of an enriched category

Fix a monoidal category $\mathbf{V} = (\mathbf{V}, \otimes, I)$.

Idea: Notion of size for $\mathbf{V}$-objects $\mapsto$ notion of size for $\mathbf{V}$-categories.

Suppose we have a semiring R and a monoid homomorphism

$$|\cdot|\colon (\operatorname{ob}\mathbf{V}/\cong, \otimes, I) \longrightarrow (R, \cdot, 1).$$

(E.g. $\mathbf{V} = \mathbf{FinSet}$, $R = \mathbb{Q}$, $|\cdot| =$ cardinality.)

Let $\mathbf{A}$ be a $\mathbf{V}$-category with finite object-set. Define an $\operatorname{ob}\mathbf{A} \times \operatorname{ob}\mathbf{A}$ matrix

$$Z_{\mathbf{A}} = \big(|\mathbf{A}(a, b)|\big)_{a,b \in A}$$

over R.

Assuming $Z_{\mathbf{A}}$ is invertible, the magnitude of $\mathbf{A}$ is

$$|\mathbf{A}| = \sum_{a,b \in \mathbf{A}} (Z_{\mathbf{A}}^{-1})(a, b) \in R.$$

Ordinary categories

Let $\mathbf{V} = \mathbf{FinSet}$, $R = \mathbb{Q}$, and $|X| = \mathrm{card}(X)$.

We obtain a notion of the magnitude $|\mathbf{A}| \in \mathbb{Q}$ of a finite category $\mathbf{A}$.

Example: If $\mathbf{A}$ is discrete then $|\mathbf{A}| = \mathrm{card}(\mathrm{ob}\,\mathbf{A})$.

Example: Every small category $\mathbf{A}$ has a classifying space $B\mathbf{A} \in \mathbf{Top}$.

And under reasonable finiteness hypotheses, it is a theorem that

$$|\mathbf{A}| = \chi(B\mathbf{A}).$$

For this reason, $|\mathbf{A}|$ is also called the Euler characteristic of $\mathbf{A}$.

(So Euler characteristic arises naturally here as a notion of size,
in accordance with Schanuel's vision.)

Metric spaces

Let $\mathbf{V} = ([0,\infty], \geq)$, $\otimes = +$ and $I = 0$. Let $R = \mathbb{R}$ and $|x| = e^{-x}$.
(Why e^{-x}? So that $x \mapsto |x|$ is a monoid homomorphism).

We obtain a notion of the magnitude $|A| \in \mathbb{R}$ of a finite metric space A.

The definition extends to compact subsets $A \subseteq \mathbb{R}^n$.

It is geometrically informative. For example:

Theorem (Meckes): Let $A \subseteq \mathbb{R}^n$ be compact. The asymptotic growth of the function $t \mapsto |tA|$ is equal to the Minkowski dimension of A.

Conjecture (with Willerton): Let $A \subseteq \mathbb{R}^2$ be compact convex. For $t > 0$,

$$|tA| = \chi(A) + \frac{\text{perimeter}(A)}{4} \cdot t + \frac{\text{area}(A)}{2\pi} \cdot t^2.$$

Linear categories

Let $\mathbf{V} = \mathbf{FDVect}$, $R = \mathbb{Q}$, and $|X| = \dim X$.

We obtain a notion of the magnitude $|\mathbf{A}|$ of a finite linear category.

Our main theorem will provide an example. . .

2. Algebraic background

Conventions

Throughout, we fix:

- an algebraically closed field K;
- a finite-dimensional associative K-algebra A (unital, but maybe not commutative).

We will consider finite-dimensional A-modules.

The atoms of the module world

Question: Which A-modules deserve to be thought of as 'atomic'?

Answer 1: The simple modules.
(A module is simple if it is nonzero and has no nontrivial submodules.)

Some facts about simple modules:

- There are only finitely many (up to isomorphism), and they are finite-dimensional.
- If S and T are simple then

$$\mathrm{Hom}_A(S, T) \cong \begin{cases} K & \text{if } S \cong T \\ 0 & \text{otherwise} \end{cases}$$

(using the assumption that K is algebraically closed).

The atoms of the module world

Question: Which A-modules deserve to be thought of as 'atomic'?

Answer 2: The projective indecomposable modules.

Some facts about projective indecomposables:

- There are only finitely many (up to isomorphism), and they are finite-dimensional.
  So the linear category **ProjIndec**(A) of projective indecomposable modules is essentially finite.
- The A-module A is a direct sum of projective indecomposable modules. Every projective indecomposable appears at least once in this sum.
- The linear category **ProjIndec**(A) has the same representations as the algebra A. That is,

$$[\mathbf{ProjIndec}(A), \mathbf{Vect}] \simeq A\text{-}\mathbf{Mod}$$

  where $[-,-]$ denotes the category of linear functors.

The atoms of the module world

How do these two answers compare?

$$\text{Simple} \;\; \substack{\not\Rightarrow \\ \not\Leftarrow} \;\; \text{projective indecomposable.}$$

But there is a natural bijection

$$\{\text{simple modules}\}/\cong \quad \longleftrightarrow \quad \{\text{projective indecomposables}\}/\cong$$

given by $S \leftrightarrow P$ iff S is a quotient of P. (It is not an equivalence of cats!)

Choose representative families

$(S_i)_{i\in I}$ of the iso classes of simple modules,
$(P_i)_{i\in I}$ of the iso classes of projective indecomposable modules,

with S_i a quotient of P_i. Then

$$\mathrm{Hom}_A(P_i, S_j) \cong \begin{cases} K & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Ext and the Euler form

For each $n \geq 0$, we have the functor

$$\mathrm{Ext}_A^n \colon A\text{-}\mathbf{Mod}^{\mathrm{op}} \times A\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}.$$

Assume now that A has finite global dimension, i.e. $\mathrm{Ext}_A^n = 0$ for all $n \gg 0$.

For finite-dimensional modules X and Y, define

$$\chi_A(X, Y) = \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(X, Y) \in \mathbb{Z}.$$

This χ_A is the Euler form of A. It is biadditive.

Crucial fact: $\boxed{\chi_A(P_i, S_j) = \delta_{ij}}$.

Also, writing $S = \bigoplus_{i \in I} S_i$,

$$\chi_A(S, S) = \sum_{i,j \in I} \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(S_i, S_j).$$

And although $\mathrm{Hom}_A(S_i, S_j)$ is trivial, $\mathrm{Ext}_A^n(S_i, S_j)$ is interesting.

3. The theorem

Statement of the theorem (again)

Recall: A is an algebra, of finite dimension and finite global dimension, over an algebraically closed field. We write:

- **ProjIndec**(A) for the linear category of projective indecomposable A-modules, which is essentially finite;
- χ_A for the Euler form of A;
- $|\mathbf{A}|$ for the magnitude of an (enriched) category $\mathbf{A}$;
- $S = \bigoplus_{i \in I} S_i$, where $(S_i)_{i \in I}$ is a representative family of the isomorphism classes of simple A-modules.

Theorem $|\mathbf{ProjIndec}(A)| = \chi_A(S, S)$.

Explicitly, this means: define a matrix

$$Z_A = \big(\dim \mathrm{Hom}_A(P_i, P_j)\big)_{i,j \in I}.$$

Then

$$\sum_{i,j \in I} (Z_A^{-1})(i,j) = \sum_{i,j \in I} \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(S_j, S_i).$$

Example: Koszul algebras (Stroppel)

Let A be a Koszul algebra.

Then A is naturally graded, and $S = A_0$.

Hence

$$|\mathbf{ProjIndec}(A)| = \sum_{n=0}^{\infty} (-1)^n \dim \operatorname{Ext}_A^n(A_0, A_0).$$

This was the first known case of the theorem.

Example: path algebras

Let $(Q_1 \rightrightarrows Q_0)$ be a finite acyclic quiver.

Take its path algebra A, which is of finite dimension and global dimension.

The simple and projective indecomposable modules are indexed by the vertex-set Q_0:

- P_i is the submodule of A spanned by the paths beginning at i
- its unique maximal submodule N_i is spanned by the *nontrivial* paths
- $S_i = P_i/N_i$ is therefore one-dimensional.

Computing with long exact sequences for Ext, we end up with

$$\chi_A(S, S) = |Q_0| - |Q_1|.$$

On the other hand, consider the magnitude of $\mathbf{ProjIndec}(A)$:

- each path from j to i induces a map $P_i \longrightarrow P_j$
- every map $P_i \longrightarrow P_j$ is a unique linear combination of such
- so $Z_{ij} = \dim \mathrm{Hom}_A(P_i, P_j)$ is the number of paths from j to i in Q.

So in this case, the theorem states that $\sum_{i,j}(Z^{-1})_{ij} = |Q_0| - |Q_1|$.

4. The proof

The Grothendieck group

The Grothendieck group $K(A)$ is the abelian group generated by all finite-dimensional A-modules, subject to

$$Y = X + Z$$

whenever

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is a short sequence. It follows that more generally,

$$\sum_{r=1}^{n} (-1)^r X_r = 0$$

in $K(A)$ whenever

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow 0$$

is an exact sequence.

Euler on Grothendieck

Previously, we defined $\chi_A(X, Y) \in \mathbb{Z}$ for any finite-dimensional A-modules X and Y.

In fact, χ_A is a well-defined bilinear form on $K(A)$: e.g. given a SES

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

and another finite-dimensional module V, we have

$$\chi_A(V, Y) = \chi_A(V, X) + \chi_A(V, Z).$$

(Proof: use the LES for $\mathrm{Ext}^*_A(V, -)$.)

Two bases for the Grothendieck group

The family (S_i) of simple modules spans $K(A)$.

Proof: for any finite-dimensional A-module X, we may take a composition series

$$0 = X_n < \cdots < X_1 < X_0 = X,$$

and then $X = \sum_r X_{r-1}/X_r$ in $K(A)$.

The family (P_i) of projective indecomposable modules also spans $K(A)$.

Proof: for any finite-dimensional A-module X, we may take a projective resolution

$$0 \longrightarrow Q_N \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow X \longrightarrow 0,$$

and then $X = \sum_r (-1)^{r+1} Q_r$ in $K(A)$. On the other hand, each Q_r is a sum of indecomposables, which are projective since Q_r is.

Both (S_i) and (P_i) are $\mathbb{Z}$-linear bases for $K(A)$.

Proof: χ_A is bilinear and $\chi_A(P_i, S_j) = \delta_{ij}$.

Proof of the theorem

We prove that the inverse of the matrix

$$Z = \Big(\dim \operatorname{Hom}_A(P_i, P_j)\Big)_{i,j\in I} = \Big(\dim \chi_A(P_i, P_j)\Big)_{i,j\in I}$$

is the matrix

$$E = \Big(\chi_A(S_j, S_i)\Big)_{i,j\in I}.$$

It will follow that $|\textbf{ProjIndec}(A)| = \sum_{i,j}(Z^{-1})_{ij} = \chi_A(S, S)$.

Proof Since (P_i) and (S_i) are both bases of $K(A)$ (over $\mathbb{Z}$), there is an invertible matrix C such that

$$P_j = \sum_{k\in I} C_{kj} S_k, \qquad S_j = \sum_{k\in I} (C^{-1})_{kj} P_k$$

in $K(A)$ for all $j \in I$.

Applying $\chi_A(P_i, -)$ to the first equation gives $\chi_A(P_i, P_j) = C_{ij}$, i.e. $Z = C$.

Applying $\chi_A(-, S_i)$ to the second equation gives $E = C^{-1}$. QED.

Conclusion

What is the right definition of the Euler characteristic of an algebra A?

A category theorist's answer:

- Schanuel taught us: Euler characteristic is the canonical measure of size.
- There is a general definition of the magnitude/Euler characteristic/size of an enriched category.
- An important enriched category associated with A is **ProjIndec**(A).
- So, define the Euler characteristic of A as the magnitude of **ProjIndec**(A).

An algebraist's answer:

- We know the importance of the Euler *form* of A, defined by a homological formula: $\chi_A(-,-) = \sum(-1)^n \dim \mathrm{Ext}_A^n(-,-)$.
- We know the importance of the simple modules, and their direct sum S.
- So, define the Euler characteristic of A as $\chi_A(S,S)$.

The theorem states that the two answers are the same.