Fourier Analysis 1
Preparing the ground

The first few weeks' lectures have two purposes: to outline some of the main themes of Fourier analysis, and to begin to prepare the ground for the precise development of the theory. Many of the questions on this sheet are not about Fourier analysis specifically, but most of the results that you are asked to prove will be needed later on.

The deadline for handing this work in is **4pm on Thursday 30 January 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

To ensure full marks, you need to do all the questions. Write in full sentences; marks will be awarded for communication and presentation. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. A function $f\colon \mathbb{R} \to \mathbb{C}$ is **1-periodic** if for all $x \in \mathbb{R}$ we have $f(x+1) = f(x)$.

   Given a real number κ, let $e_\kappa\colon \mathbb{R} \to \mathbb{C}$ be the function defined by $e_\kappa(x) = e^{2\pi i\kappa x}$.

   For which $\kappa \in \mathbb{R}$ is e_κ 1-periodic, and for which is it not? (Give proofs.)

2. *For this question only, don't worry about whether infinite sums converge.*

   (i) Show that for any double sequence $(c_k)_{k=-\infty}^{\infty}$ of complex numbers, the sum
$$\sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx} \tag{$*$}$$
   can be rewritten as
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nx) + \sum_{n=1}^{\infty} b_n \sin(2\pi nx) \tag{$\dagger$}$$
   for some sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=1}^{\infty}$ of complex numbers. Conversely, show that for any sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, the expression ($\dagger$) can be rewritten as ($*$) for some double sequence $(c_k)_{k=-\infty}^{\infty}$. Write down an equation for c_k in terms of the a_ns and b_ns.

   (ii) Now let $f\colon \mathbb{R} \to \mathbb{C}$ be a 1-periodic function whose restriction to $[0,1]$ is integrable. For $k \in \mathbb{Z}$, put
$$c_k = \int_0^1 f(x)e^{-2\pi ikx}\,dx$$
   (the kth Fourier coefficient of f), and for $n \in \mathbb{N}$, put
$$a_n = 2\int_0^1 f(x)\cos(2\pi nx)\,dx, \qquad b_n = 2\int_0^1 f(x)\sin(2\pi nx)\,dx$$
   (the classical Fourier coefficients of f, which you met in PAA). Show that (c_k), (a_n) and (b_n) satisfy the equation you wrote down at the end of (i).

   This establishes the relationship between the original sine/cosine form of Fourier series and the more elegant exponential form. From now on, we work solely with the latter.

3. In this question, you may assume that the class of integrable functions $[0,1) \to \mathbb{C}$ has the following two properties: (A) if f and g are integrable then so is $f+g$; (B) if $f\colon [0,1) \to \mathbb{C}$ is integrable then for any continuous function $\phi\colon \mathbb{C} \to \mathbb{C}$, the composite $\phi \circ f$ is integrable.

   Using these properties of integration *alone*, prove that if f and g are integrable then so is their product $f \cdot g$.

4. Let $I \subseteq \mathbb{R}$ be a bounded interval containing more than one point. Let $h\colon I \to \mathbb{R}$ be an integrable function such that $h(t) \geq 0$ for all $t \in I$ and $\int_I h(t)\,dt = 0$. Prove that $h(x) = 0$ for all $x \in I$ such that h is continuous at x. Then give an example to show that we cannot drop the hypothesis that h is continuous at x.

5. Show that there is no continuous bounded function $\delta\colon [-1/2, 1/2) \to \mathbb{R}$ with the following property: for all continuous bounded functions $f\colon [-1/2, 1/2) \to \mathbb{R}$,

$$\int_{-1/2}^{1/2} f(x)\delta(x)\,dx = f(0).$$

   This shows that there is no such thing as the 'delta function' (at least, not if it's supposed to be continuous). To make precise the intuitive idea of 'delta function', the concept of function needs to be generalized. This leads to the concept of distribution.

6. Let $I \subseteq \mathbb{R}$ be a bounded interval. For integrable functions $f, g\colon I \to \mathbb{C}$, define

$$\langle f, g\rangle = \int_I f(x)\overline{g(x)}\,dx.$$

   (i) Verify that for integrable functions $f, g, h\colon I \to \mathbb{C}$ and $a, b \in \mathbb{C}$, we have:

$$\langle f, f\rangle = \|f\|_2^2, \qquad \langle g, f\rangle = \overline{\langle f, g\rangle},$$
$$\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle, \qquad \langle f, ag + bh\rangle = \overline{a}\langle f, g\rangle + \overline{b}\langle f, h\rangle.$$

   (ii) For the rest of this question, let $f, g\colon I \to \mathbb{C}$ be integrable functions. Prove that

$$\int_I \int_I |f(x)g(y) - f(y)g(x)|^2\,dx\,dy = 2\Big(\|f\|_2^2\|g\|_2^2 - |\langle f, g\rangle|^2\Big).$$

   (Hint: when handling the modulus of a complex number z, it is often more graceful to use the formula $|z|^2 = z\overline{z}$ than to split z into its real and imaginary parts.)

   (iii) Deduce the integral version of the **Cauchy–Schwarz inequality**:

$$|\langle f, g\rangle| \leq \|f\|_2\|g\|_2.$$

   (iv) Deduce further that

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$$

   (one case of **Minkowski's inequality**). *(Hint: use $\|h\|_2 = \sqrt{\langle h, h\rangle}$.)*

Fourier Analysis 2

Periodic functions and their Fourier coefficients

The deadline for handing in this work is **4pm on Thursday 13 February 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

To ensure full marks, you need to do all the questions. Write in full sentences; marks will be awarded for communication and presentation. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. Write down:

   (i) three of the most important definitions

   (ii) three of the most important theorems

   (iii) three points you don't entirely understand

   from lectures so far.

2. A function $f\colon \mathbb{R} \to \mathbb{C}$ is **even** if $f(x) = f(-x)$ for all $x \in \mathbb{R}$, and **real** if $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. A double sequence $(c_k)_{k=-\infty}^{\infty}$ of complex numbers is **even** if $c_k = c_{-k}$ for all $k \in \mathbb{Z}$, and **real** if $c_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$.

   Let $f\colon \mathbb{R} \to \mathbb{C}$ be an integrable 1-periodic function. Write $\hat{f}$ for the double sequence $(\hat{f}(k))_{k=-\infty}^{\infty}$.

   (i) Prove that if f is real then $\hat{f}(-k) = \overline{\hat{f}(k)}$ for all $k \in \mathbb{Z}$.

   (ii) Prove that if f is even then $\hat{f}$ is even.

   (iii) Prove that if f is real and even then $\hat{f}$ is real and even.

3. True or false? (You do not need to write out any justification.)

   (i) Let $f, g\colon [0, 1) \to \mathbb{C}$ be functions with $f = g$ a.e. If f is continuous then so is g.

   (ii) Let $f, g\colon [0, 1) \to \mathbb{C}$ be functions with $f = g$ a.e. If f is integrable then so is g.

   (iii) Let $f, g\colon [0, 1) \to \mathbb{C}$ be integrable functions with $f = g$ a.e. Then $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$.

   (iv) Let $f, f_1, f_2, \ldots : [0, 1) \to \mathbb{C}$ be integrable functions. If $f_n \to f$ in $\| \cdot \|_2$ and each f_n is continuous then f is continuous.

   (v) Let $f, g\colon [0, 1) \to \mathbb{C}$ be integrable functions with $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. Then $f = g$.

   (vi) A function $\mathbb{T} \to \mathbb{C}$ is integrable if and only if the corresponding function $[0, 1) \to \mathbb{C}$ is integrable.

   (vii) A function $\mathbb{T} \to \mathbb{C}$ is continuous if and only if the corresponding function $[0, 1) \to \mathbb{C}$ is continuous.

   (viii) Let $f_1, f_2, \ldots, f, g\colon \mathbb{T} \to \mathbb{C}$ be integrable functions. If $f_n \to f$ in $\| \cdot \|_\infty$ then $\langle f_n, g\rangle \to \langle f, g\rangle$.

4. Let $f\colon \mathbb{R} \to \mathbb{C}$ be a function. Prove that:

   (i) f is continuous if and only if $f(\cdot + t) \to f$ pointwise as $t \to 0$.

   (ii) f is uniformly continuous if and only if $f(\cdot + t) \to f$ in $\| \cdot \|_\infty$ as $t \to 0$.

5. Let $f, g\colon \mathbb{T} \to \mathbb{C}$ be integrable functions, and let $k \in \mathbb{Z}$. Prove that $\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$.

6. The Dirichlet kernel D_n is defined, in our usual notation, by $D_n = \sum_{k=-n}^{n} e_k$. Prove that for all $n \geq 0$ and $t \in \mathbb{T}$,

$$D_n(t) = \begin{cases} \frac{\sin((2n+1)\pi t)}{\sin(\pi t)} & \text{if } t \neq 0 \\ 2n+1 & \text{if } t = 0. \end{cases}$$

Hint: use the fact that $e_k = e_1^k$. *Then sum the geometric series.*

7. *This question is intended to help your understanding of convolution. It will not be assessed.*

   (i) Visit http://mathoverflow.net/questions/5892 and read the answers to the question asked there: 'What is convolution, intuitively?'

   (ii) Some photo editing software (such as GIMP) includes a tool called 'convolve', used for softening sharp edges. Why do you think it might be called that? What do you think the algorithm might be?

8. *This question is intended to be much harder, and will not be assessed. Don't sink too much time into it until you've finished the rest.*

   (i) Prove that for any $a < b$ in $\mathbb{R}$, the interval $[a, b]$ does not have measure zero.

   (ii) Deduce that a subset of $\mathbb{R}$ with measure zero has empty interior (that is, contains no open subset of $\mathbb{R}$ apart from $\emptyset$).

   (iii) Show by example that a subset of $\mathbb{R}$ with empty interior need not have measure zero, even if it is compact.

   Moral: measure zero is a stronger condition than empty interior.

Fourier Analysis 3
Convergence theorems

The deadline for handing in this work is **4pm on Thursday 6 March 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

To ensure full marks, you need to do all the questions. Write in full sentences; marks will be awarded for communication and presentation. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. Write down:

   (i) three of the most important definitions

   (ii) three of the most important theorems

   (iii) three points you don't entirely understand

   from lectures so far.

2. Let D_n denote the nth Dirichlet kernel, and put $F_n = \frac{1}{n+1}(D_0 + \cdots + D_n)$ (the nth **Fejér kernel**). Prove that the sequence $(D_n(1/2))_{n=0}^{\infty}$ does not converge, but the sequence $(F_n(1/2))_{n=0}^{\infty}$ does.

   In lectures, we will see that the Dirichlet kernels behave rather wildly. We will also see that the Fejér kernels are a smoothed-out version of the Dirichlet kernels, and are easier to work with. This question offers a taste of that phenomenon.

3. (i) Prove that $\left(n\chi_{[-1/2n,1/2n)}\right)_{n=1}^{\infty}$ is a positive approximation to delta.

   (ii) Is $(D_n)_{n=0}^{\infty}$ a positive approximation to delta? Does $f * D_n \to f$ in $\|\cdot\|_2$ for all integrable functions $f\colon \mathbb{T} \to \mathbb{C}$? Justify your answers.

4. The main theorem of Part B of lectures was that for any integrable $f\colon \mathbb{T} \to \mathbb{C}$, we have $S_n f \to f$ in $\|\cdot\|_2$ and $\|\cdot\|_1$. Outline the strategy of the proof in about half a page, in either prose or a diagram.

5. (i) *It's an easy fact that for $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$ for all $x, y \in \mathbb{R}$. Put another way, $xy = \frac{1}{4}\sum_{p=0}^{1}(-1)^p|x + (-1)^p y|^2$. You'll now prove a complex analogue of this equation.*

   Show that for $w, z \in \mathbb{C}$,

   $$w\bar{z} = \frac{1}{4}\sum_{p=0}^{3} i^p |w + i^p z|^2.$$

   (There's no doubt that you can do this; the challenge is to do it elegantly!)

   (ii) Deduce from (i) and Parseval's theorem that for all integrable functions $f, g\colon \mathbb{T} \to \mathbb{C}$,

   $$\int_{\mathbb{T}} f(x)\overline{g(x)}\,dx = \sum_{k=-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}.$$

6. We write $C^n(\mathbb{T})$ for the set of functions $\mathbb{T} \to \mathbb{C}$ such that the corresponding 1-periodic function $g\colon \mathbb{R} \to \mathbb{C}$ is n times continuously differentiable (that is, $g^{(n)}$ exists and is continuous).

   (i) Let $f \in C^1(\mathbb{T})$. Prove that

   $$\widehat{f'}(k) = 2\pi i k \hat{f}(k)$$

   for all $k \in \mathbb{Z}$. Where did you use the hypothesis that f is *continuously* differentiable?

(ii) Deduce that whenever $n \geq 0$, $f \in C^n(\mathbb{T})$ and $k \in \mathbb{Z}$,

$$\widehat{f^{(n)}}(k) = (2\pi i k)^n \hat{f}(k).$$

(iii) Show that $\sup_{k \in \mathbb{Z}} |\hat{g}(k)| \leq \|g\|_1$ for all integrable $g \colon \mathbb{T} \to \mathbb{C}$.

(iv) Let $n \geq 0$, $f \in C^n(\mathbb{T})$, and $0 \neq k \in \mathbb{Z}$. Prove that

$$|\hat{f}(k)| \leq \frac{\|f^{(n)}\|_1}{(2\pi)^n} \cdot \frac{1}{|k|^n}.$$

(Moral: the smoother a function is, the faster its Fourier coefficients decay.)

(v) We will prove in lectures that for continuous functions $f \colon \mathbb{T} \to \mathbb{C}$, if $\sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty$ then $S_n f \to f$ uniformly. Assuming this, show that if $f \in C^2(\mathbb{T})$ then $S_n f \to f$ uniformly.

(Later, we will prove a stronger theorem: if $f \in C^1(\mathbb{T})$ then $S_n f \to f$ uniformly.)

7. *This question is just for fun. It will not be assessed.*

(i) In the spirit of Example B6.1, perform non-rigorous calculations to 'prove' that $1 + 2 + 3 + \cdots = -\frac{1}{12}$.

(ii) Look up the Riemann zeta function and find a fact that justifies this conclusion.

Fourier Analysis 4

Uniform and pointwise convergence of Fourier series

The deadline for handing in this work is **4pm on Thursday 20 March 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

To ensure full marks, you need to do all the questions. Write in full sentences; marks will be awarded for communication and presentation. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. Write down:

   (i) three of the most important definitions

   (ii) three of the most important theorems

   (iii) three points you don't entirely understand

   from recent lectures.

2. Define a 1-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ by
$$f(x) = \sin(2\pi x) + \cos(2\pi x)$$
($x \in \mathbb{R}$). In which (if any) of our usual five senses does the Fourier series of f converge to f?

3. Let $f\colon \mathbb{T} \to \mathbb{C}$ be an integrable function, and let $t \in \mathbb{T}$. Derive a formula for the Fourier coefficients of the function $f(\cdot + t)$ in terms of those of f.

4. Prove the Weierstrass approximation theorem (Corollary C4.5). That is, let $I \subseteq \mathbb{R}$ be a closed bounded interval, let $f\colon I \to \mathbb{C}$ be a continuous function, and let $\varepsilon > 0$. Prove that there exists a polynomial p (with complex coefficients) such that $\sup_{x\in I} |f(x) - p(x)| < \varepsilon$.

   This one is harder. Hints: rescale so that $|I| < 1$, then extend f to a continuous 1-periodic function on $\mathbb{R}$. Use a theorem about density from lectures to find a trigonometric polynomial g approximating f in $\|\cdot\|_\infty$. Then use some complex analysis to approximate g by a polynomial p.

5. Define $g\colon [-1/2, 1/2) \to \mathbb{C}$ by
$$g(x) = \begin{cases} e^{-x^2} & \text{if } x \leq -1/4, \\ \frac{|\sin 2\pi x|}{2i + \cos 2\pi x} & \text{if } x > -1/4. \end{cases}$$
Does $(S_n g)(1/3) \to g(1/3)$ as $n \to \infty$? Justify your answer.

6. Let $\alpha \in \mathbb{Q}$. By Example C6.2(ii) and Corollary C6.5, it is *not* the case that
$$\text{for all } k \in \mathbb{Z} \setminus \{0\}, \quad \frac{1}{n} \sum_{j=0}^{n} e^{2\pi i k j \alpha} \to 0 \text{ as } n \to \infty.$$
Show this directly (without using any results from lectures).

7. (i) Let $S = \{j \geq 1 : \text{the first digit of } j\pi \text{ after the decimal point is } 8\}$. Prove that $\frac{1}{n} \cdot \#(S \cap \{1, \ldots, n\}) \to 1/10$ as $n \to \infty$.

   (ii) Let $T = \{j \geq 1 : \text{the 123rd digit of } j\pi \text{ after the decimal point is } 8\}$. Prove that $\frac{1}{n} \cdot \#(T \cap \{1, \ldots, n\}) \to 1/10$ as $n \to \infty$.

(iii) Let $U = \{j \geq 1 :$ the 123rd digit of $j\pi$ after the decimal point is 8, the 124th is 9, and the 125th is $2\}$. Prove that $\frac{1}{n} \cdot \#(U \cap \{1, \ldots, n\}) \to 1/1000$ as $n \to \infty$.

(iv) Let $V = \{d \geq 1 :$ the dth digit of π after the decimal point is $8\}$. Does Weyl's equidistribution theorem imply that $\frac{1}{n} \cdot \#(V \cap \{1, \ldots, n\}) \to 1/10$ as $n \to \infty$? Justify your answer.

Fourier Analysis 5

Fourier analysis on finite abelian groups

The deadline for handing in this work is **4pm on Thursday 3 April 2014**. **Please hand it in to the MTO.** Details of how the work will be assessed can be found in the FAQ on the course Learn page.

To ensure full marks, you need to do all the questions. Write in full sentences; marks will be awarded for communication and presentation. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. Write down:

   (i) three of the most important definitions

   (ii) three of the most important theorems

   (iii) three points you don't entirely understand

   from recent lectures.

2. Let G be a finite abelian group. Let $e\colon G \to \mathbb{C}$ be a function satisfying $e(1) = 1$ and $e(xy) = e(x)e(y)$ for all $x, y \in G$. Prove that e is a character of G.

3. Let G_1 and G_2 be finite abelian groups. In this question, you will prove that $\widehat{G_1 \times G_2} \cong \widehat{G_1} \times \widehat{G_2}$.

   (i) Let $e_1 \in \widehat{G_1}$ and $e_2 \in \widehat{G_2}$. Define a function $\iota(e_1, e_2)\colon G_1 \times G_2 \to \mathbb{C}$ by
   $$\big(\iota(e_1, e_2)\big)(x_1, x_2) = e_1(x_1) \cdot e_2(x_2)$$
   ($x_1 \in G_1, x_2 \in G_2$). Prove that $\iota(e_1, e_2)$ is a character of $G_1 \times G_2$.

   (ii) By (i), ι defines a function
   $$\begin{array}{rccc} \iota\colon & \widehat{G_1} \times \widehat{G_2} & \longrightarrow & \widehat{G_1 \times G_2}, \\ & (e_1, e_2) & \longmapsto & \iota(e_1, e_2). \end{array}$$
   Prove that ι is an isomorphism of groups.

4. Let G be a finite abelian group.

   (i) Show that there is a function $\delta\colon G \to \mathbb{C}$ with the following property: for all functions $f\colon G \to \mathbb{C}$,
   $$\int_G f(x)\delta(x)\,dx = f(1).$$
   (Contrast Sheet 1, q.5. In the world of finite abelian groups, the delta function exists!)

   (ii) The convolution $f * g\colon G \to \mathbb{C}$ of functions $f, g\colon G \to \mathbb{C}$ is defined by
   $$(f * g)(x) = \int_G f(t)g(t^{-1}x)\,dt$$
   ($x \in G$). Prove that δ is an identity for convolution.

   (iii) Prove that $\widehat{f * g} = \hat{f} \cdot \hat{g}$ for all $f, g\colon G \to \mathbb{C}$.

   (iv) Find $\hat{\delta}$.

5. Let G be a finite abelian group.

   (i) Calculate $\hat{e}$, for any character e of G.

   (ii) Let $x \in G$. Define $\delta_x \colon G \to \mathbb{C}$ by $\delta_x(y) = 0$ for $y \neq x$, and $\delta_x(x) = \#G$. Calculate $\widehat{\delta_x}$.

   (iii) The integration operator $I \colon \widehat{G} \to \mathbb{C}$ is defined by $I(e) = \int_G e(x)\,dx$ for all $e \in \widehat{G}$. Find the unique function $f \colon G \to \mathbb{C}$ such that $\hat{f} = I$.

6. *Here we connect Fourier analysis on finite groups with Fourier analysis on the circle.*

   In this question, we will view the cyclic group C_n as the subgroup $\{0, 1/n, \ldots, (n-1)/n\}$ of the additive group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given a function $f \colon \mathbb{T} \to \mathbb{C}$, denote by $f^{[n]} \colon C_n \to \mathbb{C}$ the restriction of f to C_n. As usual, write e_k for the kth character of $\mathbb{T}$.

   Let $f \colon \mathbb{T} \to \mathbb{C}$ be an integrable function and $k \in \mathbb{Z}$. Prove that

   $$\hat{f}(k) = \lim_{n\to\infty} \widehat{f^{[n]}}\left(e_k^{[n]}\right).$$

   The Fourier coefficients of a function on the circle (that is, a periodic function) can therefore be computed by approximating the circle by finite cyclic groups. This tactic is used in numerical methods for computing Fourier series.