The n-Category Café
A group blog on math, physics and philosophy

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Magnitude Homology

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[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

I'm excited that **over on this thread** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#comments] , Mike Shulman has proposed a very plausible theory of magnitude homology. I think his creation could be really important! It's general enough that it can be applied in lots of different contexts, meaning that lots of different kinds of mathematician will end up wanting to use it.

However, the story of magnitude homology has so far only been told in that comments thread, which is very long, intricately nested, and probably only being followed by a tiny handful of people. And because I think this story deserves a really wide readership, I'm going to start afresh here and explain it from the beginning.

Magnitude is a *numerical* invariant of enriched categories [https://golem.ph.utexas.edu/category/2011/06/the_magnitude_of_an_enriched_c.html] . Magnitude homology is an *algebraic* invariant of enriched categories. The Euler characteristic of magnitude homology is magnitude, and in that sense, magnitude homology is a categorification of magnitude. Let me explain!

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

I'll explain twice: a short version, then a long version. After that, there's a section going into some of the details that I wanted to keep tucked out of the way. Choose the level of detail you want!

So that I don't have to keep saying it, almost everything here that's new is due to Mike Shulman, who put these ideas together **on the other thread** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#comments] . Some aspects were present in work that Aaron Greenspan did during his master's year with me (2014–15); **you can read his MSc thesis here** [http://www.maths.ed.ac.uk/~tl/docs/Greenspan_MSc_Thesis.pdf] . But Aaron and I didn't get very far, and it was Mike who made the decisive contributions and to whom this theory should be attributed.

The short version

I won't actually give the definition here — I'll just sketch its shape.

Let V be a semicartesian monoidal category. **Semicartesian** means that the unit object of V is terminal. **This isn't as unnatural a condition as it might seem!** [https://golem.ph.utexas.edu/category/2016/08/monoidal_categories_with_proje.html]

Let X be a small V-category ($=$ category **enriched** [https://en.wikipedia.org/wiki/Enriched_category#Definition] in V). **Small** means that the collection of objects of X is small (a set).

Let $A\colon V \to \mathrm{Ab}$ be a small functor. In this context, **small** means that A is the left Kan extension of its restriction to some small full subcategory of V. This condition holds automatically if the category V is small, as it often will be for us.

From this data, we define a sequence $(H_n(X; A))_{n\geq 0}$ of abelian groups, called the **(magnitude) homology of X with coefficients in A**. Dually, given instead a contravariant functor $A\colon V^{\mathrm{op}} \to \mathrm{Ab}$, there is a sequence $(H^n(X; A))_{n\geq 0}$ of **cohomology groups**. But we'll concentrate on homology.

As for any notion of homology, we can attempt to form the Euler characteristic

$$\chi(X; A) = \sum_{n\geq 0} (-1)^n \mathrm{rank}(H_n(X; A)).$$

Depending on X and A, it may or may not be possible to make sense of this infinite sum.

Examples:

- When $V = \text{Set}$ and A is chosen suitably, we recover the notion of homology and Euler characteristic of an ordinary category. What do "homology" and "Euler characteristic" mean for an ordinary category? There are several equivalent answers; one is that they're just the homology and Euler characteristic of the topological space associated to the category, called its geometric realization or classifying space. The Euler characteristic of a category is **also called** [https://golem.ph.utexas.edu/category/2011/06/the_magnitude_of_an_enriched_c.html] its magnitude.

- When V is the poset $(\mathbb{N} \cup \{\infty\}, \geq)$, made monoidal by taking $\otimes$ to be addition, graphs can be understood as special V-categories. By choosing suitable values of A, we obtain **Hepworth and Willerton's** [https://golem.ph.utexas.edu/category/2015/05/categorifying_the_magnitude_of.html] **magnitude homology of a graph** [https://arxiv.org/abs/1505.04125] . Its Euler characteristic is the **magnitude** [http://golem.ph.utexas.edu/category/2014/01/the_magnitude_of_a_graph.html] **of a graph** [http://arxiv.org/abs/1401.4623] .

- When V is the poset $([0, \infty], \geq)$, made monoidal by taking $\otimes$ to be addition, metric spaces can be understood as special V-categories. By choosing suitable values of A, we obtain a new notion of the magnitude homology of a metric space. Subject to convergence issues that haven't been fully worked out yet, its Euler characteristic is the **magnitude** [http://www.math.illinois.edu/documenta/vol-18/27.html] **of a** [https://golem.ph.utexas.edu/category/2011/01/magnitude_of_metric_spaces_a_r.html] **metric space** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html] .

The long version

Again, let's start by fixing a semicartesian monoidal category V. I'll use the letter ℓ for a typical object of V, because an important motivating case is where $V = [0, \infty]$, and in that case the objects of V are thought of as *l*engths.

> **Aside** Actually, you can be a bit more general and work with an arbitrary monoidal category V equipped with an augmentation, **as described here** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c051059] , or you can do something **more general still** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c051244] . But I'll stick with the simpler hypothesis of semicartesianness.

Step 1 Let X be a small V-category. We define a kind of nerve $N(X)$. The nerve of an ordinary category is a *single* simplicial set, but for us $N(X)$ will be a functor $V^{\text{op}} \to \text{sSet}$ into the category sSet of simplicial sets. For $\ell \in V$, the simplicial set $N(X)(\ell)$ is defined by

$$N(X)(\ell)_n = \coprod_{x_0, \ldots, x_n \in X} V\big(\ell, X(x_0, x_1) \otimes \cdots \otimes X(x_{n-1}, x_n)\big)$$

($n \geq 0$). The degeneracy maps are given by inserting identities. The inner face maps are given by composition. The outer face maps are defined using the unique maps from the first factor $X(x_0, x_1)$ and the last factor $X(x_{n-1}, x_n)$ to the unit object of V. (There are unique such maps because V is semicartesian.)

Mike wrote MS instead of N. I guess he intended the M to stand for magnitude and the S to stand for simplicial. I'm using N because I want to emphasize that it's a kind of nerve. Still, half of me regrets removing the notation MS from a construction described by Mike Shulman.

Steps 2 and 3 Let $C(X)$ be the composite functor

$$V^{\text{op}} \xrightarrow{N(X)} \text{sSet} \xrightarrow{\mathbb{Z}\cdot -} \text{sAb} \longrightarrow \text{Ch}.$$

Here sAb is the category of simplicial abelian groups, Ch is the category of chain complexes of abelian groups, and the functor $\mathbb{Z} \cdot - : \text{sSet} \to \text{sAb}$ is induced by the free abelian group functor $\mathbb{Z} \cdot - : \text{Set} \to \text{Ab}$. The unlabelled functor $\text{sAb} \to \text{Ch}$ sends a simplicial abelian group to either its unnormalized chain complex or its normalized chain complex. It won't matter which we use, for reasons I'll explain in the details section below.

Notice that $C(X)$ isn't a *single* chain complex; it's a functor into the category of chain complexes. There's one chain complex $C(X)(\ell)$ for each object ℓ of V.

Step 4 Now we bring in the other piece of data: a small functor $A: V \to \text{Ab}$, which I'll call the **functor of**

coefficients. Actually, everything that follows makes sense in the more general context of a functor $A: V \to \mathrm{Ch}$, where Ab is thought of as a subcategory of Ch by viewing an abelian group as a chain complex concentrated in degree zero. But we don't seem to have found a purpose for that extra generality, so I'll stick with Ab.

We form the tensor product of $C(X): V^{\mathrm{op}} \to \mathrm{Ch}$ with $A: V \to \mathrm{Ab}$. By definition, this is the chain complex defined by the coend formula

$$C(X) \otimes_V A = \int^{\ell \in V} C(X)(\ell) \otimes A(\ell).$$

The tensor product on the *right*-hand side is the **tensor product of chain complexes** [https://ncatlab.org/nlab/show/tensor+product+of+chain+complexes]. Under our assumption that $A(\ell)$ is concentrated in degree zero, its nth component is simply $C(X)(\ell)_n \otimes A(\ell)$.

Explicitly, this coend is the coproduct over all $\ell \in V$ of the chain complexes $C(X)(\ell) \otimes A(\ell)$, quotiented out by one relation for each map $\ell \to \ell'$ in V. Which relation? Well, given such a map, you can write down two maps from $C(X)(\ell') \otimes A(\ell)$ to the coproduct I just mentioned, and the relation states that they're equal.

This coend exists because of the smallness assumption on A. Indeed, by definition of small functor, there exists some small full subcategory W of V such that A is the left Kan extension of $A|_W$ along the inclusion $W \hookrightarrow V$. Then $C(X)|_W \otimes_W A|_W$ exists because Ch has small colimits, and you can show that it has the defining universal property of the coend above. So $C(X) \otimes_V A$ exists and is equal to $C(X)|_W \otimes_W A|_W$.

We have now constructed from X and A a *single* chain complex $C(X) \otimes_V A$.

If you choose to use *unnormalized* chains, you can unwind the coend formula to get a simple explicit formula for $C(X) \otimes_V A$:

$$(C(X) \otimes_V A)_n = \coprod_{x_0, \ldots, x_n \in X} A\big(X(x_0, x_1) \otimes \cdots \otimes X(x_{n-1}, x_n)\big)$$

with the differential that you'd guess. (This formula does assume that A is a functor from A into Ab rather than Ch. For Ch-valued A, the formula becomes slightly more complicated.) I don't think there's such a simple formula for normalized chains, at least for general V.

Step 5 The **(magnitude) homology of X with coefficients in** A, written as $H_*(X; A)$, is the homology of the chain complex $C(X) \otimes_V A$. In other words, $H_n(X; A)$ is the nth homology group of $C(X) \otimes_V A$, for $n \geq 0$.

For the definition of cohomology, let A instead be a small *contravariant* functor $V^{\mathrm{op}} \to \mathrm{Ab}$. Then we can form the chain complex

$$\mathrm{Hom}(C(X), A)_V = \int_{\ell \in V} \mathrm{Hom}(C(X)(\ell), A(\ell)).$$

The Hom on the right-hand side denotes the **closed structure** [https://ncatlab.org/nlab/show/internal+hom+of+chain+complexes] on the monoidal category of chain complexes. And $H^*(X; A)$, the **cohomology of X with coefficients in** A, is defined as the homology of the chain complex $\mathrm{Hom}(C(X), A)_V$.

Everything is functorial in the way it should be: homology $H_*(X; A)$ is covariant in X, cohomology $H^*(X; A)$ is contravariant in X, and both are covariant in the functor A of coefficients.

Example: ordinary categories

When $V = \mathrm{Set}$, a small V-category is just a small category X.

The functor $N(X): \mathrm{Set}^{\mathrm{op}} \to \mathrm{sSet}$ sends $\ell \in \mathrm{Set}$ to the ℓth power of the ordinary nerve. So, we might suggestively write $N(X)(\ell)$ as $N(X)^\ell$ instead.

Now let's think about the functor of coefficients, which is some small functor $A: \mathrm{Set} \to \mathrm{Ab}$. For A to be small means exactly that there is some small full subcategory W of Set such that A is the left Kan extension of $A|_W$ along the inclusion $W \hookrightarrow \mathrm{Set}$. For instance, choose an abelian group B and define $A(\ell)$ to be the coproduct $\ell \cdot B$ of ℓ copies of B. Then A is small, since if we take $W \subset \mathrm{Set}$ to be the full subcategory consisting of just the one-element set then A is the left Kan extension of its restriction to W. Let's write A as $- \cdot B: \mathrm{Set} \to \mathrm{Ab}$.

element set then A is the left Kan extension of its restriction to W. Let's write A as $- \cdot B$: Set $\to$ Ab.

The general definition gives us homology groups $H_*(X; - \cdot B)$ for every small category X and abelian group B. These homology groups, more normally written as $H_*(X; B)$, are actually something familiar. In simplicial terms, they're simply the homology of the ordinary nerve of X (with coefficients in B). In terms of topological spaces, they're just the homology of the geometric realization (classifying space) of X.

Example: graphs

Let $V = (\mathbb{N} \cup \{\infty\}, \geq)$, a poset seen as a category. The objects of V are the natural numbers together with ∞, there's exactly one map $\ell \to m$ when $\ell \geq m$, and there are no maps $\ell \to m$ when $\ell < m$. It's a monoidal category under addition. Any graph X can be seen as a V-category: the objects are the vertices, and $X(x, y) \in V$ is the number of edges in a shortest path from x to y (understood to be ∞ if there is no such path at all).

So, we're going to get a homology theory of graphs.

What about the coefficients? Well, the first point is that we don't have to worry about the smallness condition. The category V is small, so it's automatic that any functor on V is small too.

The second, important, point is that *every object ℓ of V gives rise to a functor* $\delta_\ell: V \to \mathrm{Ab}$, defined by

$$\delta_\ell(m) = \begin{cases} \mathbb{Z} & \text{if } m = \ell, \\ 0 & \text{if } m \neq \ell. \end{cases}$$

($m \in V$). We're going to use δ_ℓ as our functor of coefficients.

So, for any graph X and natural number ℓ, we get homology groups $H_*(X; \delta_\ell)$. It turns out that $H_n(X; \delta_\ell)$ is exactly what Richard Hepworth and Simon Willerton **called** [https://arxiv.org/abs/1505.04125] the magnitude homology group $\mathrm{MH}_{n,\ell}(X)$.

I'll repeat Richard and Simon's definition here, so that you can see concretely what Mike's general theory actually produces in a specific situation. Let X be a graph. For integers $n, \ell \geq 0$, let $\mathrm{MC}_{n,\ell}(X)$ be the free abelian group on the set

$$\{(x_0, \ldots, x_n) \in X^{n+1} : x_0 \neq x_1 \neq \cdots \neq x_n,\ d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell\}.$$

For $1 \leq i \leq n-1$, define $\partial_i: \mathrm{MC}_{n,\ell}(X) \to \mathrm{MC}_{n-1,\ell}(X)$ by

$$\partial_i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if } d(x_{i-1}, x_{i+1}) = d(x_{i-1}, x_i) + d(x_i, x_{i+1}), \\ 0 & \text{otherwise.} \end{cases}$$

Then define $\partial: \mathrm{MC}_{n,\ell}(X) \to \mathrm{MC}_{n-1,\ell}(X)$ by $\partial = \sum_{i=1}^{n-1} (-1)^i \partial_i$. This gives a chain complex $\mathrm{MC}_{*,\ell}(X)$ for each natural number ℓ. The **Hepworth–Willerton magnitude homology** group $\mathrm{MH}_{n,\ell}(X)$ is defined to be its nth homology.

So, this two-case formula for the differential, involving the triangle inequality, somehow comes out of Mike's general definition. I'll explain how in the details section below.

Incidentally, Richard and Simon proved a Künneth theorem, an excision theorem and a Mayer–Vietoris theorems for their magnitude homology of graphs. Can these be generalized magnitude homology of arbitrary enriched categories?

Example: metric spaces

Let V be the poset $([0, \infty], \geq)$, made into a monoidal category in the same way that $\mathbb{N} \cup \{\infty\}$ was. As **Lawvere pointed out long ago** [https://golem.ph.utexas.edu/category/2014/02/metric_spaces_generalized_logi.html], any metric space can be seen as a V-category.

So, we get a homology theory of metric spaces. More exactly, we have a graded abelian group $H_*(X; A)$ for each metric space X and functor $A: [0, \infty] \to \mathrm{Ab}$. Exactly as for graphs, every element $\ell \in [0, \infty]$ gives rise to a functor $\delta_\ell: [0, \infty] \to \mathrm{Ab}$, taking value $\mathbb{Z}$ at ℓ and 0 elsewhere. So we get a group $H_n(X; \delta_\ell)$ for each $n \in \mathbb{N}$ and $\ell \in [0, \infty]$.

Explicitly, this group $H_n(X, \delta_\ell)$ turns out to be the same as the group $\mathrm{MH}_{n,\ell}(X)$ that you get from Hepworth and Willerton's definition above by simply crossing out the word "graph" and replacing it by "metric space", and letting ℓ range over $[0, \infty]$ rather than $\mathbb{N} \cup \{\infty\}$.

letting ℓ range over $[0, \infty]$ rather than $\mathbb{N} \cup \{\infty\}$.

But here's the thing. There are some metric spaces, including most finite ones, where the triangle inequality is *never* an equality (except in the obvious trivial situations). For such spaces, the Hepworth–Willerton differential ∂ is always 0. Hence the homology groups are the same as the chain groups, which tend to be rather large. For instance, that's almost always the case when X is a random finite collection of points in Euclidean space. So homology fails to do its usual job of summarizing useful information about the space.

In that situation, we might prefer to use different coefficients. So, let's think again about the construction of the functor $\delta_\ell : V \to \mathrm{Ab}$ from the object $\ell \in V$. This construction makes sense for any partially ordered set V, and it also makes sense not only for *single elements* (objects) of V, but arbitrary *intervals* in V.

What I mean is the following. An **interval** J in a poset V is a subset with the property that if $\ell_1 \leq \ell_2 \leq \ell_3$ in V with $\ell_1, \ell_3 \in J$ then $\ell_2 \in J$. For any interval $J \subseteq V$, there's a functor $\delta_J : V \to \mathrm{Ab}$ defined on objects by

$$\delta_J(\ell) = \begin{cases} \mathbb{Z} & \text{if } \ell \in J, \\ 0 & \text{otherwise.} \end{cases}$$

It's defined on maps by sending everything to either a zero map or the identity on $\mathbb{Z}$. For instance, if J is a trivial interval $\{\ell\}$ then δ_J is the functor δ_ℓ that we met before.

I observed a few paragraphs back that when X is a finite metric space, $H_*(X; \delta_\ell)$ typically isn't very interesting. However, it seems likely that $H_*(X; \delta_J)$ is more interesting for nontrivial intervals $J \subseteq [0, \infty]$. The **idea** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c051173] is that it introduces some blurring, to compensate for the fact that the triangle inequality is never *exactly* an equality. And here we get into territory that seems close to that of **persistent homology** [https://www.math.upenn.edu/~ghrist/preprints/barcodes.pdf] ... but this connection still needs to be explored!

Decategorification: from homology to magnitude

For any homology theory of any kind of object X, we can attempt to define the Euler characteristic of X as the alternating sum of the ranks of the homology groups. We immediately have to ask whether that sum makes sense.

It may be that only finitely many of the homology groups are nontrivial, in which case there's no problem. Or it may be that infinitely many of the groups are nontrivial, but the Euler characteristic can be made sense of using one or other technique for **summing divergent series** [https://en.wikipedia.org/wiki/Divergent_series] . Or, it may be that the sum is beyond salvation. Typically, if you want the Euler characteristic to make sense — or even just in order for the ranks to be finite — you'll need to impose some sort of finiteness condition on the object that you're taking the homology of.

The idea — perhaps the *entire point* of magnitude homology — is that its Euler characteristic should be equal to magnitude. For some enriching categories V, we have a theorem saying exactly that. For others, we don't... but we do have some formal calculations suggesting that there's a theorem waiting to be found. We haven't got to the bottom of this yet.

I'll say something about the general situation, then I'll explain the state of the art in the three examples above.

In general, for a semicartesian monoidal category V, a small V-category X, and a small functor $A : V \to \mathrm{Ab}$, we want to define the **Euler characteristic of X with coefficients in A** as

$$\chi(X; A) = \sum_{n \geq 0} (-1)^n \mathrm{rank}(H_n(X; A)).$$

Here's how it looks in our three running examples: categories, graphs and metric spaces.

- In the case $V = \mathrm{Set}$, we're talking about the Euler characteristic of a category X. Take $A = - \cdot \mathbb{Z}$, as defined **above** [#ordinary] . Then the homology group $H_n(X; A)$ is equal to $H_n(X; \mathbb{Z})$, the nth homology of the category X with coefficients in $\mathbb{Z}$. That's the same as the nth homology of the nerve (or its geometric realization).

  To make sense of $\chi(X; \mathbb{Z})$, we impose a finiteness condition. Assume that the category X is finite, skeletal, and contains no nontrivial endomorphisms. Then the nerve of X has only finitely many nondegenerate simplices, from which it follows that only finitely many of the homology groups are nontrivial. So, the sum is finite and $\chi(X; \mathbb{Z})$ makes sense.

  Under these finiteness hypotheses, what actually *is* $\chi(X; \mathbb{Z})$? Since $H_n(X; \mathbb{Z})$ is the nth homology of the nerve of X with integer coefficients, $\chi(X;\mathbb{Z})$ is the ordinary (simplicial/topological) Euler characteristic of the nerve of X. And it's a **theorem** [http://www.math.uni-bielefeld.de/documenta/vol-13/02.html] that this is equal to the Euler characteristic of the category X, **defined combinatorially** [http://www.math.uni-bielefeld.de/documenta/vol-13/02.html] and also called the "magnitude" of X.

So for a small category X, the Euler characteristic of the magnitude homology $H_*(X; -\cdot\mathbb{Z})$ is indeed the magnitude of X. In other words: **magnitude homology categorifies magnitude**.

- Take a graph X, seen as a category enriched in $V = (\mathbb{N}\cup\{\infty\},\ \geq\ )$. For each natural number ℓ, we can try to define the Euler characteristic

  $$\chi(X;\delta_\ell) = \sum_{n\geq 0} (-1)^n \mathrm{rank}(H_n(X;\delta_\ell)).$$

  I said earlier that these homology groups are the same as Hepworth and Willerton's homology groups $\mathrm{MH}_{n,\ell}(X)$, and I described them explicitly.

  To make sure that the ranks are all finite, let's assume that the graph X is finite. That alone is enough to guarantee that the sum defining $\chi(X;\delta_\ell)$ is finite. Why? Well, from the definition of the chain groups $\mathrm{MC}_{n,\ell}(X)$, it's clear that $\mathrm{MC}_{n,\ell}(X)$ is trivial when $n > \ell$. Hence the same is true of $\mathrm{MH}_{n,\ell}(X)$, which means that the sum defining $\chi(X;\delta_\ell)$ might as well run only from $n=0$ to $n=\ell$.

  At the moment, our graph has not *one* Euler characteristic but an *infinite sequence* of them:

  $$\chi(X;\delta_0),\ \chi(X;\delta_1),\ \chi(X;\delta_2),\ \cdots$$

  Let's assemble them into a single formal power series over $\mathbb{Z}$:

  $$\chi(X) := \sum_{\ell\in\mathbb{N}} \chi(X;\delta_\ell) q^\ell = \sum_{n\geq 0}(-1)^n \sum_{\ell\in\mathbb{N}} \mathrm{rank}(H_n(X;\delta_\ell)) q^\ell,$$

  where q is a formal variable. (You might wonder what's happened to $\ell=\infty$. In principle, it should be present in the sum. However, if we adopt the convention that $q^\infty = 0$ then it might as well not be. It will become clear when we look at metric spaces that this *is* the right convention to adopt.)

  On the other hand, viewing graphs as enriched categories leads to the notion of the **magnitude of a graph** [https://arxiv.org/abs/1401.4623] . The magnitude of a finite graph X is a formal expression in a variable q, and can be understood either as a rational function in q or as a power series in q. Hepworth and Willerton **showed** [https://arxiv.org/abs/1505.04125] that the power series $\chi(X)$ above is precisely the magnitude of X, seen as a power series.

  So in the case of graphs too, **magnitude homology categorifies magnitude**.

- Finally, consider a metric space X, viewed as a category enriched in $V = ([0,\infty],\ \geq\ )$. For each $\ell\in V$, we want to define

  $$\chi(X;\delta_\ell) = \sum_{n\geq 0} (-1)^n \mathrm{rank}(H_n(X;\delta_\ell)).$$

  I have no idea what these homology groups look like when X is a familiar geometric object such as a disk or line, so I don't know how often these ranks are finite. But they're certainly finite if X has only finitely many points, so let's assume that.

  The sum on the right-hand side is, then, automatically finite. To see this, the argument is almost the same as for graphs. For graphs, we used the fact that the distance between two distinct vertices is always at least 1, from which it followed that the homology groups $H_n(X;\delta_\ell)$ can only be nonzero when $n\leq\ell$. Now in a finite metric space, distances can of course be less than 1, but finiteness implies that there's a minimal nonzero distance: η, say. Then $H_n(X;\delta_\ell)$ can only be nonzero when $n\leq \ell/\eta$. That's why the sum is finite.

  We've now assigned to our metric space not one Euler characteristic but a one-parameter family of them. That is, we've got an integer $\chi(X,\delta_\ell)$ for each $\ell\in[0,\infty]$. Actually, all but countably many of these integers are zero. Better still, for each real m there are only finitely many $\ell\leq m$ such that $\chi(X,\delta_\ell)\neq 0$. (I'll explain why in the details section.) So, it's not too crazy to write down the formal expression

$$\chi(X) = \sum_{\ell\in[0,\infty)} \chi(X;\delta_\ell) q^\ell .$$

There are a couple of ways to think about the expression on the right-hand side. You can treat q as a formal variable and the expression as a **Hahn** [https://en.wikipedia.org/wiki/Hahn_series] **series** [https://ncatlab.org/nlab/show/Hahn+series] (like a power series, but with non-integer real powers allowed). Or you can (attempt to) evaluate at a particular value of q in $\mathbb{R}$ or $\mathbb{C}$ or some other setting where the sum makes analytic sense.

So far no one knows how exactly we should proceed from here, but it looks as if the story goes something like this.

Remember, we're trying to show that magnitude homology categorifies magnitude, which in this instance means that $\chi(X)$ should be equal to the magnitude of a metric space X. That's a real number, and it's defined in terms of negative exponentials e^{-d} of distances d, so let's put $q = e^{-1}$. (This explains why we can ignore $\ell = \infty$, since then $q^\ell = e^{-\infty} = 0$.) I'm not claiming that anything converges! You can treat e^{-1} as a formal variable for the time being, although at *some* stage we'll want to interpret it as an actual real number.

It's a useful little lemma that when you have a bounded chain complex C, the alternating sum of the ranks of the groups C_n is equal to the alternating sum of the homology groups $H_n(C)$. So,

$$\chi(X;\delta_\ell) = \sum_{n\geq 0} (-1)^n \mathrm{rank}(\mathrm{MC}_{n,\ell}(X))$$

where MC denotes the Hepworth–Willerton chain groups that I defined earlier. Substituting this into the definition of $\chi(X)$ gives

$$\chi(X) = \sum_{n\geq 0} (-1)^n \sum_{\ell\in[0,\infty)} \mathrm{rank}(\mathrm{MC}_{n,\ell}(X)) e^{-\ell}.$$

That's potentially a doubly infinite sum. But we can do some *formal* calculations leading to the conclusion that $\chi(X)$ is indeed equal to the magnitude of the metric space X (that is, the sum of all the entries of the inverse of the matrix $(e^{-d(x,y)})_{x,y\in X}$). Again, that's deferred to the details section below. It's not clear how to make rigorous sense of it, but I'm confident that it can somehow be done.

So, **magnitude homology categorifies magnitude** in all three of our examples… well, definitely in the first two cases, and tentatively in the third. Of course, we'd like to make a *general* statement to the effect that homology categorifies magnitude over an arbitrary base category V. The metric space case illustrates some of the difficulties that we might expect to encounter in making a general statement.

Details and proofs

The rest of this post mostly consists of supporting details that we figured out in the **other thread** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html] . I've mostly only bothered to include the points that weren't immediately obvious to us (or me, at least).

If you've read this far, bravo! You can think of what follows as an appendix.

From simplicial abelian groups to chain complexes

The relationship between simplicial abelian groups and chain complexes is a classical part of homological algebra, but there's at least one aspect of it that some of us in the **old thread** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#comments] hadn't previously appreciated.

First, the definitions. Let G be a simplicial abelian group. The **unnormalized chain complex** $C(G)$ is defined by $C_n(G) = G_n$, the differentials being the alternating sums of the face maps. The degenerate elements of $C_n(G)$ generate a subgroup $D_n(G)$, which assemble to give a subcomplex of $C(G)$. The **normalized chain complex** is $C(G)/D(G)$.

Now here are two facts. First, there's an isomorphism of chain complexes $C(G) \cong D(G) \oplus \frac{C(G)}{D(G)}$, natural in G. Second, the projection and inclusion maps between $C(G)$ and $C(G)/D(G)$ are mutually inverse up to a chain homotopy *that is natural in G* (in the obvious sense). That naturality will be crucial for us. We therefore say that $C(G)$ and $C(G)/D(G)$ are **naturally chain homotopy equivalent**.

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The functoriality of the nerve construction

Given a monoidal category V and a small V-category X, we defined a functor

$$N(X) \colon V^{\text{op}} \to \text{sSet}$$

by

$$N(X)(\ell)_n = \coprod_{x_0, \ldots, x_n \in X} V\big(\ell, X(x_0, x_1) \otimes \cdots \otimes X(x_{n-1}, x_n)\big).$$

Obviously N is functorial in X: any V-functor $F \colon X \to Y$ induces a map of simplicial sets $N(F)_\ell \colon N(X)(\ell) \to N(Y)(\ell)$ for each $\ell \in V$, and this map is natural in ℓ.

Less obvious is that N is functorial in the following 2-dimensional sense. Take V-functors

$$F, G \colon X \rightrightarrows Y$$

and a V-natural transformation $\alpha \colon F \to G$. The claim is that for each $\ell \in V$, there's an induced simplicial homotopy α_ℓ from $N(F)_\ell$ to $N(G)_\ell$. Moreover, α_ℓ is natural in ℓ.

How does this work? I'm pretty much a klutz with things simplicial, so let me explain it in a concrete way and refer to **this comment of Mike's** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c051215] for a more abstract perspective.

Fix ℓ. We have our two maps

$$N(F)_\ell, N(G)_\ell \colon N(X)(\ell) \rightrightarrows N(Y)_\ell.$$

By definition, a **simplicial homotopy** from $N(F)_\ell$ to $N(G)_\ell$ is a map $h \colon N(X)(\ell) \times \Delta^1 \to N(Y)(\ell)$ of simplicial sets that satisfies the appropriate boundary conditions. Here Δ^1 means the representable simplicial set $\Delta(-, [1])$. There are two maps from the terminal simplicial set $1 = \Delta^0$ to Δ^1, corresponding to the two maps $[0] \rightrightarrows [1]$ in Δ. The "boundary conditions" are that the two composites in the diagram

$$N(X)(\ell) \cong N(X)(\ell) \times 1 \rightrightarrows N(X)(\ell) \times \Delta^1 \xrightarrow{h} N(Y)(\ell)$$

are equal to $N(F)_\ell$ and $N(G)_\ell$.

Concretely, a simplicial homotopy h from $N(F)_\ell$ to $N(G)_\ell$ consists of a map of sets

$$h_\phi \colon N(X)(\ell)_n \to N(Y)(\ell)_n$$

for each map $\phi \colon [n] \to [1] = \{0, 1\}$ in Δ. When ϕ is the map with constant value 0, h_ϕ is required to be equal to $N(F)_{\ell,n}$, and when h has constant value 1, h_ϕ is required to be equal to $N(G)_{\ell,n}$. The maps h_ϕ also have to satisfy some other equations which I don't need to mention.

There are $n+2$ maps $[n] \to [1]$ in Δ, so what this means *really* explicitly is that a simplicial homotopy from $N(F)$ to $N(G)$ consists of an ordered list of $n+2$ functions $N(X)(\ell)_n \to N(Y)(\ell)_n$ for each $n \geq 0$. The first has to be $N(F)_{\ell,n}$, the last has to be $N(G)_{\ell,n}$, and the whole lot have to hang together in some reasonable way. So roughly speaking, a simplicial homotopy is a kind of discrete path between two simplicial maps, as you'd probably expect.

We're supposed to be building a simplicial homotopy from $N(F)$ to $N(G)$ out of a V-natural transformation $\alpha \colon F \to G$. So, let's recall what a V-natural transformation actually is. More or less by definition, α consists of a map

$$\alpha_{x,x'} \colon X(x, x') \to Y(F(x), G(x'))$$

in V for each $x, x' \in X$ (subject to some axioms). For instance, when $V = \mathrm{Set}$, this map sends $f \in X(x, x')$ to the diagonal of the naturality square for f.

Now let $n \geq 0$. For any objects $x_0, \ldots, x_n$ of X, we can build from F, G and α a sequence of $n+2$ maps in V, which for ease of typesetting I'll show for $n = 3$ (and you'll guess the general pattern):

$$\begin{array}{rcl}
C(x_0, x_1) \otimes C(x_1, x_2) \otimes C(x_2, x_3) & \longrightarrow & D(Fx_0, Fx_1) \otimes D(Fx_1, Fx_2) \otimes D(Fx_2, Fx_3), \\
C(x_0, x_1) \otimes C(x_1, x_2) \otimes C(x_2, x_3) & \longrightarrow & D(Fx_0, Fx_1) \otimes D(Fx_1, Fx_2) \otimes D(Fx_2, Gx_3), \\
C(x_0, x_1) \otimes C(x_1, x_2) \otimes C(x_2, x_3) & \longrightarrow & D(Fx_0, Fx_1) \otimes D(Fx_1, Gx_2) \otimes D(Gx_2, Gx_3), \\
C(x_0, x_1) \otimes C(x_1, x_2) \otimes C(x_2, x_3) & \longrightarrow & D(Fx_0, Gx_1) \otimes D(Gx_1, Gx_2) \otimes D(Gx_2, Gx_3), \\
C(x_0, x_1) \otimes C(x_1, x_2) \otimes C(x_2, x_3) & \longrightarrow & D(Gx_0, Gx_1) \otimes D(Gx_1, Gx_2) \otimes D(Gx_2, Gx_3).
\end{array}$$

These $n+2$ maps in V induce, in the obvious way, $n+2$ maps of sets

$$\coprod_{x_0, \ldots, x_n \in X} V(\ell, C(x_0, x_1) \otimes \cdots \otimes C(x_{n-1}, x_n)) \longrightarrow \coprod_{y_0, \ldots, y_n \in Y} V(\ell, D(y_0, y_1) \otimes \cdots \otimes D(y_{n-1}, y_n))$$

for each $\ell \in V$. The domain and codomain here are just $N(X)(\ell)_n$ and $N(Y)(\ell)_n$: so we have $n+2$ maps $N(X)(\ell)_n \to N(Y)(\ell)_n$. The first of these maps is $N(F)_{\ell,n}$ and the last is $N(G)_{\ell,n}$. Some checking reveals that these maps, taken over all n, do indeed determine a simplicial homotopy from $N(F)_\ell$ to $N(G)_\ell$. Moreover, everything is obviously natural in ℓ. So that's our natural simplicial homotopy!

Functoriality of the tensor product

Let $A\colon V \to \mathrm{Ch}$ be a small functor. For any functor $B\colon V^{\mathrm{op}} \to \mathrm{Ch}$, we can form the tensor product

$$B \otimes_V A = \int^{\ell \in V} B(\ell) \otimes A(\ell),$$

which is a chain complex. Obviously this determines a functor

$$- \otimes_V A\colon [V^{\mathrm{op}}, \mathrm{Ch}] \to \mathrm{Ch}.$$

A little less obviously, $- \otimes_V A$ transforms any natural chain homotopy into a chain homotopy.

In other words, take functors $P, Q\colon V^{\mathrm{op}} \rightrightarrows \mathrm{Ch}$ and natural transformations $\kappa, \lambda\colon P \rightrightarrows Q$. (So, κ and λ consist of chain maps $\kappa_\ell, \lambda_\ell\colon P(\ell) \rightrightarrows Q(\ell)$ for each $\ell \in V$, natural in ℓ.) Suppose we also have a chain homotopy $h_\ell\colon \kappa_\ell \to \lambda_\ell$ for each ℓ, and that h_ℓ is natural in ℓ. The claim is that there's an induced chain homotopy $h \otimes_V A$ between the chain maps

$$\kappa \otimes_V A,\ \lambda \otimes_V A\colon P \otimes_V A \rightrightarrows Q \otimes_V A.$$

To show this, the key point is that a chain homotopy between the chain maps $\kappa_\ell, \lambda_\ell\colon P(\ell) \to Q(\ell)$ can be understood as a chain map $P(\ell) \otimes \mathrm{I} \to Q(\ell)$ satisfying appropriate boundary conditions. Here I (for "interval") is the chain complex

$$\cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0 \to 0 \to \cdots$$

with the two copies of $\mathbb{Z}$ in degrees 0 and 1. Once you adopt this viewpoint, it's straightforward to prove the claim, using only the associativity of $\otimes$ and the fact that $\otimes$ distributes over colimits.

An important consequence is that if two functors $B, B'\colon V^{\mathrm{op}} \rightrightarrows \mathrm{Ch}$ are naturally chain homotopy equivalent, then the complexes $B \otimes_V A$ and $B' \otimes_V A$ are chain homotopy equivalent.

It doesn't matter whether you normalize your chains

Let X be a small V-category. The functor $C(X)\colon V^{\mathrm{op}} \to \mathrm{Ch}$ was defined by first building from X a certain functor $\mathbb{Z} \cdot N(X)\colon V^{\mathrm{op}} \to \mathrm{sAb}$, then turning simplicial sets into chain complexes. I (or rather Mike) said that it doesn't matter whether you do that last step with unnormalized or normalized chains. Why not?

Earlier in this "details" section, I recalled the fact that the two chain complexes coming from a simplicial abelian group G are not only chain homotopy equivalent, but chain homotopy equivalent in a way that's natural in G. We can apply this fact to the simplicial abelian group $\mathbb{Z} \cdot N(X)(\ell)$, for each $\ell \in V$. It implies that the two chain complexes coming from $\mathbb{Z} \cdot N(X)(\ell)$ are chain homotopy equivalent naturally in ℓ. Or, said another way, the two versions of $C(X): V^{\mathrm{op}} \to \mathrm{Ch}$ that you get by choosing the "unnormalized" or "normalized" option are *naturally* chain homotopy equivalent.

But we just saw that when two functors $B, B': V^{\mathrm{op}} \rightrightarrows \mathrm{Ch}$ are naturally chain homotopy equivalent, their tensor products with A are chain homotopy equivalent. So, the two versions of $C(X)$ have the same tensor product with A, up to chain homotopy equivalence. In other words, the chain homotopy equivalence class of $C(X) \otimes_V A$ is unaffected by which version of $C(X)$ you choose to use. The homology $H_*(C; A)$ of that chain complex is, therefore, also unaffected by this choice.

Invariance of magnitude homology under equivalence of categories

It's a fact that the magnitude of an enriched category is invariant not only under equivalence, but even under the existence of an adjunction (at least, if both magnitudes are well-defined). Something similar is true for magnitude *homology*, as follows.

Let $F, G: X \rightrightarrows Y$ be V-functors between small V-categories. We'll show that if there exists a V-natural transformation from F to G then the maps

$$H_*(X; A) \rightrightarrows H_*(Y; A)$$

induced by F and G are equal (for any coefficients A). It will follow that whenever you have V-categories that are equivalent, or even just connected by an adjunction, their homologies are isomorphic. (Even "adjunction" can be weakened further, but I'll leave that as an exercise.)

The proof is mostly a matter of assembling previous observations. Take a V-natural transformation $\alpha: F \to G: X \to Y$. We have functors

$$N(X), N(Y): V^{\mathrm{op}} \rightrightarrows \mathrm{sSet},$$

natural transformations

$$N(F), N(G): N(X) \rightrightarrows N(Y),$$

and (as we saw previously) a natural simplicial homotopy from $N(F) \to N(G)$ induced by α. When we pass from simplicial sets to chain complexes, this natural simplicial homotopy turns into a natural chain homotopy (Lemma 8.3.13 of Weibel's book). So, the natural transformations $C(F)$ and $C(G)$ between the functors

$$C(X), C(Y): V^{\mathrm{op}} \rightrightarrows \mathrm{Ch}$$

are naturally chain homotopic. It follows from another of the previous observations that the chain maps

$$C(F) \otimes_V A, \;\; C(G) \otimes_V A : \; C(X) \otimes_V A \rightrightarrows C(Y) \otimes_V A$$

are chain homotopic. Hence they induce the same map $H_*(X; A) \to H_*(Y; A)$ on homology, as claimed.

Homology of graphs and of metric spaces

Earlier, I claimed that Mike's general theory of homology of enriched categories reproduces Richard Hepworth and Simon Willerton's theory of magnitude homology of graphs, by choosing the coefficients suitably. It's trivial to extend Richard and Simon's theory from graphs to metric spaces, as I did earlier; and I claimed that this too is captured by the general theory.

I'll prove this now in the case of metric spaces. It will then be completely clear how it works for graphs.

Let X be a metric space, seen as a category enriched in $V = [0, \infty]$. Let $\ell \geq 0$ be a real number, and recall the functor $\delta_\ell: V \to \mathrm{Ab}$ from earlier. The aim is to show that the groups $H_n(X, \delta_\ell)$ and $\mathrm{MH}_{n,\ell}(X)$ are isomorphic, where the latter is defined à la Hepworth–Willerton.

The nerve functor $N(X): V^{\mathrm{op}} \to \mathrm{sSet}$ is given by

$$N(X)(\ell')_n = \{(x_0,\ldots,x_n)\colon d(x_0,x_1)+\cdots+d(x_{n-1},x_n)\le \ell'\}.$$

The unnormalized chain group $C(X)(\ell')_n$ is simply the free abelian group on this set, but in order to make the connection with Richard and Simon's definition, we're going to use the *normalized* version of $C(X)$. It's not too hard to see that the normalized $C(X)(\ell')_n$ is the free abelian group on the set

$$\{(x_0,\ldots,x_n)\colon x_0\ne x_1\ne\cdots\ne x_n,\ d(x_0,x_1)+\cdots+d(x_{n-1},x_n)\le \ell'\}$$

with differentials $\partial=\sum_{i=0}^n(-1)^i\partial_i$, where

$$\partial_i(x_0,\ldots,x_n)=\begin{cases}(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n) & \text{if } i=0 \text{ or } i=k \text{ or } x_{i-1}\ne x_{i+1},\\ 0 & \text{otherwise.}\end{cases}$$

Now we have to compute $C(X)\otimes_V\delta_\ell$. I know two ways to do this. You can use the definition of coend directly, as **Mike does here** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c051059]. Alternatively, note that for any functor of coefficients $A\colon V\to \mathrm{Ab}$,

$$\begin{aligned}(C(X)\otimes_V A)_n &= \int^{\ell'\in V}(C(X)(\ell')\otimes A(\ell'))_n\\ &= \int^{\ell'\in V}C(X)(\ell')_n\otimes A(\ell')\\ &= \int^{\ell'\in V}\coprod_{x_0\ne\cdots\ne x_n}\mathbb{Z}\cdot V(\ell',d(x_0,x_1)+\cdots+d(x_{n-1},x_n))\otimes A(\ell')\\ &= \coprod_{x_0\ne\cdots\ne x_n}\int^{\ell'\in V}\mathbb{Z}\cdot V(\ell',d(x_0,x_1)+\cdots+d(x_{n-1},x_n))\otimes A(\ell')\\ &= \coprod_{x_0\ne\cdots\ne x_n}A(d(x_0,x_1)+\cdots+d(x_{n-1},x_n)),\end{aligned}$$

where the last step is by the density formula. We're interested in the case $A=\delta_\ell$, and then the expression $A(\cdots)$ in the last line is either $\mathbb{Z}$ if the distances sum to ℓ, or 0 if not. So

$$(C(X)\otimes_V\delta_\ell)_n=\{(x_0,\ldots,x_n)\colon x_0\ne x_1\ne\cdots\ne x_n,\ d(x_0,x_1)+\cdots+d(x_{n-1},x_n)=\ell\}.$$

That's exactly Richard and Simon's chain group $\mathrm{MC}_{n,\ell}(X)$. With a little more thought, you can see that the differentials agree too. Thus, the chain complexes $C(X)\otimes_V\delta_\ell$ and $\mathrm{MC}_{*,\ell}(X)$ are isomorphic. It follows that their homologies are isomorphic, as claimed.

Decategorification for metric spaces

The final stretch of this marathon post is devoted to finite metric spaces — specifically, how the magnitude of a finite metric space can be obtained as the Euler characteristic of its magnitude homology. Here's where there are some gaps.

Let X be a finite metric space. For each $\ell\in V$, we have the Euler characteristic

$$\chi(X;\delta_\ell)=\sum_{n\ge 0}(-1)^n\mathrm{rank}(H_n(X;\delta_\ell)).$$

The ranks here are finite because the sets $N(X)(\ell)_n$ are manifestly finite. We saw earlier that the sum itself is finite, but let me repeat the argument slightly more carefully. First, these homology groups are the same as the Hepworth–Willerton homology groups. Second, the Hepworth–Willerton chain groups $\mathrm{MC}_{n,\ell}(X)$ are trivial when $n>\ell/\eta$, where η is the minimum nonzero distance occurring in X. So, the same is true of the homology groups $\mathrm{MH}_{n,\ell}(X)=H_n(X;\delta_\ell)$.

Let $\mathbb{L}_X\subseteq[0,\infty]$ be the set of (extended) real numbers occurring as finite sums $d(x_0,x_1)+\cdots+d(x_{n-1},x_n)$ of distances in X. Although this set is usually infinite, it's always countable. Better still, $\mathbb{L}_X\cap[0,L]$ is finite for all real $L\ge 0$. It's easy to prove this, again using the fact that there's a minimum nonzero distance.

For a number ℓ that's *not* in $\mathbb{L}_X$, the Hepworth–Willerton chain groups $\mathrm{MC}_{*,\ell}(X)$ are trivial, so the homology groups $\mathrm{MH}_{*,\ell}(X) = H_*(X;\delta_\ell)$ are trivial too. Hence $\chi(X;\delta_\ell) = 0$. Or in other words: $\chi(X;\delta_\ell)$ only stands a chance of being nonzero if ℓ belongs to the countable set $\mathbb{L}_X$.

So, in the definition

$$\chi(X) = \sum_{\ell\in[0,\infty]} \chi(X;\delta_\ell)e^{-\ell},$$

that scary-looking sum over all $\ell \in [0,\infty]$ might as well only be over the relatively tame range $\ell \in \mathbb{L}_X$.

Now let's do a formal calculation. Back in the main part of the post (just before the **start** [#details] of this "details" section), I observed that

$$\chi(X) = \sum_{n\geq 0} (-1)^n \sum_{\ell\in[0,\infty)} \mathrm{rank}(\mathrm{MC}_{n,\ell}(X))e^{-\ell}.$$

Now $\mathrm{MC}_{n,\ell}(X)$ is the free abelian group on the set

$$\{(x_0,\ldots,x_n) : x_0 \neq x_1 \neq \cdots \neq x_n,\ d(x_0,x_1) + \cdots + d(x_{n-1},x_n) = \ell\},$$

so $\mathrm{rank}(\mathrm{MC}_{n,\ell}(X))$ is the cardinality of this set. Hence, working formally,

$$\sum_{\ell\in[0,\infty)} \mathrm{rank}(\mathrm{MC}_{n,\ell}(X))e^{-\ell} = \sum_{x_0\neq\cdots\neq x_n} e^{-d(x_0,x_1)}e^{-d(x_1,x_2)}\cdots e^{-d(x_{n-1},x_n)}.$$

Let Z_X be the square matrix with rows and columns indexed by the points of X, and entries $Z_X(x,y) = e^{-d(x,y)}$. Write I for the $X \times X$ identity matrix, and write $\mathrm{sum}(M)$ for the sum of all the entries of a matrix M. Then

$$\sum_{x_0\neq\cdots\neq x_n} e^{-d(x_0,x_1)}\cdots e^{-d(x_{n-1},x_n)} = \mathrm{sum}((Z_X - I)^n).$$

So our earlier formula

$$\chi(X) = \sum_{n\geq 0} (-1)^n \sum_{\ell\in[0,\infty)} \mathrm{rank}(\mathrm{MC}_{n,\ell}(X))e^{-\ell}$$

now gives

$$\chi(X) = \sum_{n\geq 0} (-1)^n \mathrm{sum}(Z_X - I)^n = \mathrm{sum}\left(\sum_{n\geq 0} (I - Z_X)^n\right).$$

Again *formally speaking*, the part inside the brackets is a geometric series whose sum is Z_X^{-1}. So, the conclusion is that

$$\chi(X) = \mathrm{sum}(Z_X^{-1}).$$

The right-hand side is by definition the magnitude of the metric space X (at least, assuming that Z_X is invertible).

So, using non-rigorous formal methods, we've achieved our goal. That is, we've shown that the magnitude of a finite metric space is the Euler characteristic of its magnitude homology.

We know how to make *some* of this rigorous. The basic idea is that to sum a possibly-divergent series $\sum_{n\geq 0} (-1)^n a_n$, we "vary the value of -1" by replacing it with a formal variable t. Thus, we define the formal power series $f(t) = \sum_{n\geq 0} a_n t^n$, hope that f is formally equal to a rational function, hope that the rational function f doesn't have a pole at -1, and if not, interpret $\sum_{n\geq 0} (-1)^n a_n$ as $f(-1)$.

That's a time-honoured technique for summing divergent series. To apply it in this situation, here's a little theorem about matrices that essentially appears in a **paper by Clemens Berger and me** [http://intlpress.com/HHA/v10/n1/a3/] :

> **Theorem** *Let Z be a square matrix of real numbers. Then:*

- *The formal power series* $f(t) = \sum_{n\geq 0} \mathrm{sum}((Z-I)^n) \cdot t^n$ *is rational.*

- *If* Z *is invertible, the value of the rational function* f *at* -1 *is (defined and) equal to* $\mathrm{sum}(Z^{-1})$.

This result provides a respectable way to interpret the last part of the unrigorous argument presented above — the bit about the geometric series. But the earlier parts remain to be made rigorous.

Posted at September 6, 2016 12:04 AM UTC

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Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Incidentally, I agonized over notation.

- I called the base category V. Everyone agrees on that. (Well, if I was Latexing I'd use $\mathcal{V}$, but on the blog it's easier to stick to a plain V.)

- Usually in enriched category theory, the objects of the base category V are called things like X (or x). I've used ℓ instead, for two reasons. First, I *didn't* use X and x because I wanted them for something else. Second, ℓ is what we used in earlier conversations, it's what Richard and Simon used, and in the important examples of graphs and metric spaces, it stands for **l**ength.

- Usually I'd call an enriched category something like A or C, at the opposite end of the alphabet from V. But Mike used A for the coefficients (reasonably enough), so I wanted to avoid that. He used C for the category. However, he also used C to stand for chain, and just about everyone writing on homological algebra does the same, so I wanted to avoid that. I chose X, because it's a normal kind of letter for a graph, a metric space, or generally something that you might take the homology of.

- I used N, C and H for the nerve, chain complex and homology functors. Mike used MS, MC and H, with M standing for magnitude. As I said in the post, I think it's good to use N to signal that it's a nerve construction, but I'm agnostic on whether the C and H should have Ms in front of them.

  I don't know whether Mike's homology theory should be called "magnitude homology" or simply "homology". Since magnitude homology is the categorification of homology in the same sense as Khovanov homology is the categorification of the Jones polynomial, calling it "magnitude homology" is like saying "Jones polynomial homology" (or more euphonically, "Jones homology") instead of "Khovanov homology". That would seem entirely reasonable. On the other hand, if there are no other theories of homology for enriched categories, maybe it should just be called "homology" without adornment.

  But that comes with a risk. If Mike writes this up and just calls it "homology", someone else will call it "Shulman homology" and the name will stick. Much as he'll deserve that, I'm a firm believer that descriptive names are better than named-after-people names — e.g. "Kullback–Leibler divergence" vs. "relative entropy". In particular, "magnitude homology" is better than "Shulman homology" (sorry, Mike!). To avert the possibility of the terminology heading that way, the correct tactic must be to call it magnitude homology from the start :-)

I'm writing all this here because I want everyone to use this comments thread to discuss notation and terminology rather than mathematical substance, of course.

Posted by: Tom Leinster on September 6, 2016 2:04 AM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

As for “MS”, I was just copying the notation used by **Hepworth and Willerton** [https://arxiv.org/abs/1505.04125] for the same thing in the case of graphs (Remark 44). I only realized later that it was also my initials. (-:O I’m very happy to call it N instead.

Posted by: Mike Shulman on September 6, 2016 2:07 PM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

> I’m writing all this here because I want everyone to use this comments thread to discuss notation and terminology rather than mathematical substance, of course.

Being British, I realise that you mean this literally; so with regard to your comments about ‘Shulman homology’, I note that in the post you use the term ‘Hepworth-Willerton chain groups’.

Anyway, nice “summary”.

Posted by: Simon Willerton on September 6, 2016 10:33 PM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

> nice “summary”.

Thanks! I know you’re kidding, in some respects at least; this might be the longest post I’ve ever written. But I do hope that the “short version” *does* function as a summary. Even the long version doesn’t take *that* long to get to the definition, and the pace of it was intended to be leisurely.

Re terminology named after people, I’m aware of my hypocrisy, and actually this gives me an insight into why so many things in mathematics are named after people rather than having useful descriptive names. It’s not because there are legions of mathematicians who think it’s better that way — it’s simply *easier*.

So, I want a name for the chain groups that you and Richard called $\mathrm{MC}_{*,*}$, to distinguish them from the ones in Mike’s theory. What should I call them? I lazily named them after the two of you, but as you know, I’d prefer to use a descriptive name instead. What do you suggest?

(This is a test of both your linguistic flair and your selflessness. If you don’t suggest anything good, those chain groups will go on being named after you.)

Posted by: Tom Leinster on September 6, 2016 10:51 PM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

As long as you’re trying to write something for a pretty general audience, would you define “V-category”? By analogy with “R-algebra” I would assume it’s a category equipped with a functor from V. But then you say base, so I’d think a functor to V.

Posted by: Allen Knutson on September 6, 2016 3:06 AM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Oh, sorry. “V-category” is a synonym for **“category enriched in V** [https://en.wikipedia.org/wiki/Enriched_category#Definition] ”. I’ve edited the post to say this.

Posted by: Tom Leinster on September 6, 2016 3:10 AM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Thanks Tom for such a monumental summary of those findings!

In case people from relevant fields tune in, it would be good to hear of potentially interesting magnitude homologies for enrichment by different semicartesian categories.

From the other thread, we have the possibility of enrichment in **convex spaces** [https://golem.ph.utexas.edu/category/2016/08/monoidal_categories_with_proje.html#c051162] .

And there was a risky **punt** [https://golem.ph.utexas.edu/category/2016/08/monoidal_categories_with_proje.html#c051238] on Bruhat-Tits buildings.

Posted by: David Corfield on September 6, 2016 8:40 AM | Permalink | Reply to this

Posted by: David Corfield on September 6, 2016 8:49 AM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

> monumental

Haha, yes, it’s a whopper!

Apart from wanting to tell the world about magnitude homology, there’s a secret reason why I wanted to get everything typed up *now*. The semester that’s about to start for me is very heavy on teaching and admin, and I’m going to have extremely limited time for anything else. So I wanted to make a good record of exactly where we’re at (as far as my understanding permits) in order that I can come back to it later when I’ve forgotten all the details.

Thanks for linking to those developing ideas on the **monoidal categories with projections** [https://golem.ph.utexas.edu/category/2016/08/monoidal_categories_with_proje.html#comments] thread. I’ve been reading them without contributing. It would be spectacular if the Bruhat–Tits idea came off.

Posted by: Tom Leinster on September 6, 2016 9:00 AM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

> The semester that’s about to start for me is very heavy on teaching and admin, and I’m going to have extremely limited time for anything else. So I wanted to make a good record of exactly where we’re at (as far as my understanding permits) in order that I can come back to it later when I’ve forgotten all the details.

Sounds great to me; I’m also embarking on a very busy semester, and additionally we are expecting our second baby in November.

Posted by: Mike Shulman on September 6, 2016 2:13 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Many congratulations!

Posted by: Richard Williamson on September 6, 2016 7:50 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Congratulations!

Posted by: Tom Leinster on September 6, 2016 3:54 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Thanks!

Posted by: Mike Shulman on September 6, 2016 6:12 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Given that constant drive in certain parts to homotopify everything in sight, what scope for a homology from enriching in semicartesian monoidal $(\infty, 1)$-categories?

We have a very sketchy entry at nLab for **cartesian monoidal (infinity,1)-category** [https://ncatlab.org/nlab/show/cartesian+monoidal+%28infinity,1%29-category] .

Can a semicartesian version be far away?

We also have an entry **enriched (infinity,1)-category** [https://ncatlab.org/nlab/show/enriched+%28infinity,1%29-category] . Enrichment seems to be **possible** [https://arxiv.org/abs/1312.3178] in an arbitrary monoidal ∞-category.

Posted by: David Corfield on September 6, 2016 11:50 AM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

No bites yet? With everyone heading out of the sunlit uplands of Summer research into the dark valleys of Autumn teaching, can we not just put down a marker here?

From that Gepner and Hausgeng **article** [https://arxiv.org/abs/1312.3178] I mentioned:

> despite the large amount of work that has been carried out on the foundations of ∞-category theory, above all by Joyal and Lurie, the theory is in many ways still in its infancy, and the analogues of many concepts from ordinary category theory remain to be explored. In this paper we begin to study the natural analogue in the ∞-categorical context of one such concept, namely that of *enriched categories*.
>
> our theory gives a good setting in which to develop ∞-categorical analogues of many concepts from enriched category theory, as we hope to demonstrate in future work.
>
> The theory we set up in this article is the first completely general theory of weak enrichment.

Seeing that everything tends to go through right to the $(\infty, 1)$ case when things are done properly, can we say that it's plausible that magnitude homology carries over here?

Instead of coefficients in Ab, perhaps spectra.

I see such a thing as the **Simplicial nerve of an A-infinity category** [https://arxiv.org/abs/1312.2127] is being devised.

Is there anything to compare in the $(\infty, 1)$ world with Lawvere's surprising $([0, \infty], \geq)$?

Posted by: David Corfield on September 7, 2016 10:06 AM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Wow, thank you for this very impressive writeup! I want to emphasize that the definition and its properties developed over the course of a lengthly discussion with (mainly) Tom, so he should definitely also get some credit. Richard Williamson also made some useful contributions, such as the sufficiency of natural chain homotopy and pushing me to think harder about the smallness hypothesis on A.

In particular, last night I independently performed the "unwinding" of the coend that Tom mentions above to get

$$(C(X) \otimes_V A)_n = \bigoplus_{x_0,\ldots,x_n \in X} A(X(x_0, x_1) \otimes \ldots \otimes X(x_{n-1}, x_n))$$

and noticed that this formula doesn't depend on A being a small functor! So actually, the magnitude homology can be defined for *any* functor $A: V \to \mathrm{Ab}$ (and I'm sure that $A: V \to \mathrm{Ch}$ works too).

In particular, this perspective that de-emphasizes smallness gives a different perspective on the ordinary case. The functor $- \cdot \mathbb{Z}$ that Tom used to get out ordinary homology of ordinary categories has another name: it's the free abelian group functor! So the homology of an ordinary category with $\mathbb{Z}$ coefficients is its magnitude homology with coefficients in the free abelian group functor.

This also makes for a somewhat simpler version of the generalization to non-semicartesian V. In that case the "coefficients" consist not only of $A: V \to \mathrm{Ab}$ but also $F: X \to V$ and $G: X^{\mathrm{op}} \to V$, and the nerve is replaced by a two-sided simplicial bar construction

$$B(G, X, F)(\ell)_n = V(\ell, F(x_0) \otimes X(x_0, x_1) \otimes \cdots \otimes X(x_{n-1}, x_n) \otimes G(x_n))$$

giving in place of the simple formula above

$$\bigoplus_{x_0,\ldots,x_n \in X} A(F(x_0) \otimes X(x_0, x_1) \otimes \ldots \otimes X(x_{n-1}, x_n) \otimes G(x_n))$$

If $V = \mathrm{Ab}$ and we take A to be the *identity functor*, and specialize to the case when X has one object and is thus just a ring, then I believe the homology of this simplicial abelian group gives the "relative Tor" $\mathrm{Tor}_*^{X/\mathbb{Z}}(F, G)$. So at least in that case, we get something known.

Posted by: Mike Shulman on September 6, 2016 2:04 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

I fear I'm replying too quickly to this post, but I have to start catching up on other commitments now so here goes…

We both noticed that for the unnormalized version of $C(X)$, there is an explicit formula for $C(X) \otimes_V A$ that makes sense regardless of whether A is small. So, you could write down a much shorter definition of magnitude homology (let's say in the case of a semicartesian monoidal category V):

> *The magnitude homology of a small V-category X, with coefficients in a functor $A \colon V \to \mathrm{Ab}$, is the homology of the chain complex whose nth group is*
>
> $$\bigoplus_{x_0, \ldots, x_n \in X} A\big(X(x_0, x_1) \otimes \cdots \otimes X(x_{n-1}, x_n)\big)$$
>
> *and whose differential is given in the "obvious" way.*

That's great. But at some stage we want to use normalized chains, e.g. to make the connection with Hepworth and Willerton's magnitude homology for graphs. How do we do this without the assumption of smallness? Our existing argument uses properties of the functor $- \otimes_V A \colon [V^{\mathrm{op}}, \mathrm{Ch}] \to \mathrm{Ch}$ — but no such functor exists if A is not small.

I guess it's all OK in the sense that when you unwind all the arguments sufficiently, they're just elementary algebraic manipulations that need no smallness condition. But that's not a proof!

> I'm sure that $A \colon V \to \mathrm{Ch}$ works too

By my calculation, if we use the unnormalized version of $C(X)$ then for an arbitrary small functor $A \colon V \to \mathrm{Ch}$,

$$(C(X) \otimes_V A)_n = \bigoplus_{i+j=n} \bigoplus_{x_0, \ldots, x_i \in X} A_j\big(X(x_0, x_1) \otimes \cdots \otimes X(x_{i-1}, x_i)\big).$$

> [For general monoidal category V, not necessarily semicartesian,] the "coefficients" consist not only of $A \colon V \to \mathrm{Ab}$ but also $F \colon X \to V$ and $G \colon X^{\mathrm{op}} \to V$.

I wonder whether this begins to resolves a question about coefficients that had been bothering me.

In **what I think of as the "standard" or "classical" framework** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c050956] , one takes the (co)homology of an ordinary category X with coefficients in a functor $X^{\mathrm{op}} \to \mathrm{Ab}$. (Or maybe $X \to \mathrm{Ab}$; I won't attempt to get the variance right.) However, in the magnitude homology framework as described in my post, the coefficents are a functor $\mathrm{Set} \to \mathrm{Ab}$.

So, the two approaches simply use different kind of coefficient systems. They do agree in some sense: given an abelian group B, you can form either the functor from X to Ab with constant value B or the functor $\mathrm{Set} \to \mathrm{Ab}$ that takes copowers of B, and the two homologies are then the same. But still, it puzzles me that the coefficients are of different types.

In the approach you're describing now, the magnitude homology of an ordinary category X would have coefficients in a triple of functors (A, F, G) where

$$A \colon \mathrm{Set} \to \mathrm{Ab}, \qquad F \colon X \to \mathrm{Set}, \qquad G \colon X^{\mathrm{op}} \to \mathrm{Set}.$$

We still aren't seeing a functor $X^{(\mathrm{op})} \to \mathrm{Ab}$ here! (Though you could imagine using A, F and G to build one.) What's going on?

Posted by: Tom Leinster on September 6, 2016 3:39 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

> at some stage we want to use normalized chains, e.g. to make the connection with Hepworth and Willerton's magnitude homology for graphs

Indeed. However, the calculation that unwinds the coend makes sense even *before* we make our simplicial objects into chain complexes. That is, we have a simplicial *abelian group* whose nth group is

$$\bigoplus_{x_0, \ldots, x_n \in X} A(X(x_0, x_1) \otimes \cdots \otimes X(x_{n-1}, x_n))$$

and the unnormalized $C(X) \otimes_V A$ is the unnormalized one associated to this. I suspect that the *normalized* version of $C(X) \otimes_V A$ is just the *normalized* chain complex associated to this simplicial abelian group. In fact, I think this should follow from abstract nonsense: the normalized and unnormalized chain complexes associated to a simplicial abelian group should be obtained by tensoring it with some canonical cosimplicial chain complexes, an operation which commutes with $\otimes_V$. If this is true, then we should be fine.

Posted by: Mike Shulman on September 6, 2016 8:05 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

> In what I think of as the "standard" or "classical" framework, one takes the (co)homology of an ordinary category X with coefficients in a functor $X^{\mathrm{op}} \to \mathrm{Ab}$.... However, in the magnitude homology framework as described in my post, the coefficents are a functor $\mathrm{Set} \to \mathrm{Ab}$.

Good question! I hadn't thought of that.

One thing we can do is replace F and G by a single V-functor $H\colon X^{\mathrm{op}} \otimes X \to P$, where P is a V-category with copowers, and then take A to be a functor $P \to \mathrm{Ab}$. Then the chain groups become

$$\bigoplus_{x_0,\ldots,x_n \in X} A((X(x_0,x_1) \otimes \cdots \otimes X(x_{n-1},x_n)) \odot H(x_n,x_0))$$

where $\odot$ denotes the copower $V \times P \to P$. Given F and G we take $P = V$ and $H(x,y) = G(x) \otimes F(y)$. If we instead take $H(x,y) = X(x,y)$, we get the "magnitude Hochshild homology" suggested by Richard **here** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c051237].

And in the case $V = \mathrm{Set}$, we could start with $B\colon X \to \mathrm{Ab}$ and take $P = \mathrm{Ab}$, $A = \mathrm{Id}_{\mathrm{Ab}}$, and $H(x,y) = B(y)$ (or, if we instead used $B\colon X^{\mathrm{op}} \to \mathrm{Ab}$, then $H(x,y) = B(x)$). I don't know whether this reproduces the usual homology of a category with coefficients in B, but at least it has the same input.

Posted by: Mike Shulman on September 6, 2016 8:15 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

With the appearance of 'relative Tor', I'd just like to mention, for when people come back to this story, that over on the other thread I outline a possible construction of a 'Hochschild homology/cohomology' theory as well using Mike's ideas, just using a slightly different collection of simplicial sets. Assuming that I've not made a mistake, it'd be interesting to know whether the corresponding Euler characteristic is interesting; what it detects for graphs, metric spaces, etc.

Posted by: Richard Williamson on September 6, 2016 8:01 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Perhaps this has already been said, but the nerve of a metric space reminds me of the various complexes used in the study of persistent homology. In fact, I think it's exactly the filtered Cech complex they use there.

It also occurs to me that once you've taken the nerve, you have a nice simplicial presheaf on V and for a lot of purposes, you might just stop there! You can do a lot of homotopy theory with simplicial presheaves. This would be even more interesting if V carries the structure of a site, to get a more interesting model structure on simplicial presheaves.

It seems natural to ask what are the derived functors of the homology and cohomology functors you've defined. But maybe the nerve of a V-category is automatically bifibrant in a suitable model structure on simplicial presheaves, so that these functors are "already derived"?

Another thing this suggests is that you might lift a model structure from simplicial presheaves on V to a model structure on V-categories, and then ask how it relates to other model structures on V-categories.

Posted by: Tim Campion on September 6, 2016 2:09 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

From the brief foray into it that I mentioned **here** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c050980], aren't the Cech and Rips complexes as used in persistent homology somewhat different? The Cech one looks for inhabited intersections of balls of a given radius, while the Rips looks at limiting edge lengths.

As I mentioned also there, there is ‘squeezing’ of a kind going on.

Posted by: David Corfield on September 6, 2016 2:19 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Yes, I think everyone who’s run into persistent homology feels some resonance here! It would be really fantastic if someone could make a definite connection — actually prove some theorems — and I think we’re close to the stage where that’s a real possibility.

In fact, when John started this conversation with the **question** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c050913]

> Is there any way to generalize the Hepworth–Willerton homology from graphs to general finite metric spaces?

my instant reply was

> **I would *love* it if someone found a way to do this.**
>
> Aaron Greenspan and I spent a while trying to do it ourselves, but we didn’t get too far. Then recently, I was at a fantastic **applied topology conference** [http://atmcs7.appliedtopology.org/] where all the talk of persistent homology revived my urge to do it.

My **talk** [http://www.maths.ed.ac.uk/~tl/turin/turin.pdf] at that conference was an attempt to get applied topologists interested in magnitude. The very last slide summarizes some of the connections between the two subjects that I envisaged.

Then later in that thread, **starting here** [https://golem.ph.utexas.edu/category/2016/08/a_survey_of_magnitude.html#c050975] , David started talking about persistent homology too, going into more detail — e.g. about Čech and Rips complexes. You really need signposts for a thread that long!

As he just said, there’s a difference between the Čech and Rips complexes, and page 3 of the **paper by Ghrist he linked to** [https://www.math.upenn.edu/~ghrist/preprints/barcodes.pdf] is a good source for this.

(When consulting the literature, it’s useful to know that some people say “Vietoris complex” instead of “Rips complex”. Others try to be even-handed by saying “Vietoris–Rips”. I guess there are other people still with delicate alphabetic sensibilities who say “Rips–Vietoris” — the same people who speak of “Čech–Stone compactification” :-))

Here’s a categorically pertinent point that I haven’t seen made explicitly in applied topology: the Rips complex is something associated to a *metric space*, whereas the Čech complex is something associated to a *subspace of a metric space*. In Ghrist’s paper and many other places, all the spaces concerned are embedded in $\mathbb{R}^n$, so that distinction is invisible. But in principle that’s the way it is.

Posted by: Tom Leinster on September 6, 2016 4:00 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Right, unless I’m confused, the nerve of a metric space is not the same as its Cech or Rips complex, although they all live in the same category (simplicial presheaves on $[0,\infty]$) and in each case the n-simplices are $(n+1)$-tuples $(x_0,\ldots,x_n)$ satisfying some condition — the conditions are different in each case. In $N(X)$ we require that $d(x_0,x_1)+\cdots+d(x_{n-1},x_n)\le\ell$. In the Cech complex we require that $\bigcap_i B(x_i,\ell/2)\neq\emptyset$ in some ambient metric space. And in the Rips complex we require that each $d(x_i,x_j)\le\ell$ (so in particular, the Rips complex is determined by its 1-skeleton, which apparently makes it more computationally tractable).

Posted by: Mike Shulman on September 6, 2016 6:13 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

I’m not quite sure what you mean by “derived functor” in this context. Tom explained why the magnitude homology of a category is “homotopy invariant” as an operation on the category (respects equivalences), by way of showing that the nerve itself is also homotopy invariant. The only question along these lines I can think of that isn’t answered is whether the magnitude homology of a V-category factors through its nerve by a homotopy-invariant operation on $\mathrm{sSet}^{V^{\mathrm{op}}}$, i.e. whether $H_*(X)$ depends only on the homotopy type of $N(X)$. That is an interesting question, and I agree that the answer should be yes if $N(X)$ is sufficiently cofibrant (I don’t think fibrancy is necessary, nor is it likely to happen unless we use some "quasicategory-type" model structure on $\mathrm{sSet}^{V^{op}}$).

In fact, now that I think about it, $N(X)$ looks fairly cofibrant: in each simplicial degree it is a coproduct of representables. This seems close to being projectively cofibrant, which in turn ought to be enough to make everything homotopy-invariant. Projective cell complexes of simplicial presheaves are obtained by "gluing on representable simplices", i.e. pushing out along maps of the form $\partial\Delta^n \times V(-,v) \to \Delta^n \times V(-,v)$; so we could try to construct $N(X)$ in this way by inducting up along n and at each step using all the possible values of $v = C(x_0, x_1) \otimes \cdots \otimes C(x_{n-1}, x_n)$.

I think this could only fail because of degeneracies: if some $x_i = x_{i+1}$ and a map $\ell \to C(x_0, x_1) \otimes \cdots \otimes C(x_{n-1}, x_n)$ factors through the unit $I \to C(x_i, x_{i+1})$, then it will already be present before we "glue on that cell", whereas gluing on the cell would produce another copy of it that we don't want. If the unit maps $I \to C(x,x)$ are all isomorphisms (as they must be for $V = [0,\infty]$, for instance) then we should be able to avoid this by only gluing on cells corresponding to the nondegenerate sequences with $x_i \neq x_{i+1}$ for all i. Otherwise, we still do have to glue something on, but I suspect we can do something fancier to make it work. For instance, suppose that

1. The unit maps $I \to C(x,x)$ are all monomorphisms. This is automatic if C is semicartesian, since any map out of a terminal object is mono, but I still have the general case in mind as well. Conditions like this are also fairly common when we try to do homotopy theory involving enriched categories.

2. The Day tensor product on $\mathrm{sSet}^{V^{\mathrm{op}}}$ satisfies the pushout-product axiom for monomorphisms. I don't recall seeing this condition anywhere before, nor have I thought about what sort of condition it imposes on V.

Then I suspect we can build a "degeneracies" monomorphism of presheaves $\partial C(x_0, \ldots, x_n) \to V(-, C(x_0, x_1) \otimes \cdots \otimes C(x_{n-1}, x_n))$ by repeated pushout-products of the monos $V(-, I) \to V(-, C(x_i, x_{i+1}))$ for all i such that $i = i+1$. Taking the pushout-product of this with $\partial\Delta^n \to \Delta^n$ we get a projective cofibration that we can glue on to make the degeneracies correct.

Here's another thought that I may as well throw out there: it doesn't feel quite right to me to regard $N(X)$ as a simplicial presheaf on V. I would rather regard it as a functor $\mathrm{el}\sharp X \to V$ where $\mathrm{el}\sharp X$ is the category of elements of the codiscrete simplicial set on the objects of X, in which case we can just set

$$N'(X)(x_0, \ldots, x_n) = C(x_0, x_1) \otimes \cdots \otimes C(x_{n-1}, x_n).$$

The simplicial presheaf that Tom called $N(X)$ is obtained by applying the Yoneda embedding and then left Kan extending along the discrete opfibration $\mathrm{el}\sharp X \to \Delta^{\mathrm{op}}$. But $N'(X)$ has the advantage that the "unwound coend" formulation factors through it; it's just obtained by composing with $A\colon V \to \mathrm{Ab}$ rather than the Yoneda embedding, and *then* left Kan extending to Δ^{op} (and then making a simplicial abelian group into a chain complex). This feels to me like an even more obviously "homotopy-invariant operation", except that I don't know what category $N'(X)$ lives in (as X varies) or what homotopy theory it might have.

Posted by: Mike Shulman on September 6, 2016 6:14 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Here's a cool feature of magnitude homology of metric spaces:

> ***1st magnitude homology measures lack of convexity.***

Here I'm talking about completely arbitrary metric spaces, not just finite ones.

To explain this, I'm going to use the Hepworth–Willerton approach to magnitude homology. Let X be a metric space and $\ell > 0$ a real number. The first few of the associated chain groups are

$$\begin{aligned}
&\mathrm{MC}_{0,\ell}(X) = 0, \\
&\mathrm{MC}_{1,\ell}(X) = \mathbb{Z} \cdot \{(x_0, x_1) : x_0 \neq x_1,\ d(x_0, x_1) = \ell\}, \\
&\mathrm{MC}_{2,\ell}(X) = \mathbb{Z} \cdot \{(x_0, x_1, x_2) : x_0 \neq x_1 \neq x_2,\ d(x_0, x_1) + d(x_1, x_2) = \ell\}.
\end{aligned}$$

Here I'm writing $\mathbb{Z} \cdot S$ for the free abelian group on a set S. I'm also ignoring the case $\ell = 0$. (It's trivial: $\mathrm{MC}_{n,0}(X)$ is 0 unless $n = 0$, in which case it's $\mathbb{Z} \cdot X$.) The differential

$$\partial\colon \mathrm{MC}_{1,\ell}(X) \to \mathrm{MC}_{0,\ell}(X)$$

is zero. So, its kernel is $\mathrm{MC}_{1,\ell}(X)$. The differential

$$\partial\colon \mathrm{MC}_{2,\ell}(X) \to \mathrm{MC}_{1,\ell}(X)$$

is defined on generators by

$$\partial(x_0, x_1, x_2) = \begin{cases} -(x_0, x_2) & \text{if } d(x_0, x_1) + d(x_1, x_2) = d(x_0, x_2), \\ 0 & \text{otherwise.} \end{cases}$$

Let's say that a point x_1 is **between** points x_0 and x_2 if $d(x_0, x_1) + d(x_1, x_2) = d(x_0, x_2)$, and **strictly between** if also $x_0 \neq x_1 \neq x_2$. Then the image of ∂ is generated by the pairs (x_0, x_2) such that there exists a point strictly between x_0 and x_2.

The 1st homology $H_{1,\ell}(X)$ is the quotient of the kernel just computed by the image just computed. In other words, $H_{1,\ell}(X)$ is the free abelian group on the set of pairs (x_0, x_1) such that $d(x_0, x_1) = \ell$ and there is no point strictly between x_0 and x_1.

A metric space is **Menger convex** if for any pair of distinct points there exists a point strictly between them. Our calculation immediately implies:

> **Theorem** *Let X be a metric space. Then X is Menger convex if and only if $H_{1,\ell}(X) = 0$ for all $\ell \geq 0$.*

Menger convexity looks like a rather weak condition, but it's not. In fact, let X be a metric space with the property that closed bounded subsets are compact. The following are equivalent:

- X is Menger convex.
- X is geodesic, i.e. for all $x, y \in X$, say distance D apart, there is an isometry $[0, D] \to X$ joining x and y.

(This appears as Theorem 2.6.2 of Athanase Papadopoulos's 2005 book *Metric spaces, convexity and nonpositive curvature*, though I assume it's much older than that.)

For instance:

> **Corollary** *A closed set $X \subseteq \mathbb{R}^n$ is convex if and only if $H_{1,\ell}(X) = 0$ for all $\ell \geq 0$.*

Generally, the more a space fails to be convex, the larger the groups $H_{1,\ell}(X)$ will tend to be. That's because there will be more pairs of points with no point strictly between them, and these pairs are the generators of the first homology groups.

You could be a bit more subtle and ask what happens at different length scales. For instance, consider the metric space $\mathbb{Z}$ with its usual metric. All the homology groups $H_{*,\ell}(\mathbb{Z})$ vanish unless ℓ is an integer.

- $\ell = 0$: we have $H_{1,0}(\mathbb{Z}) = 0$ (as for any space).
- $\ell = 1$: the abelian group $H_{1,1}(\mathbb{Z})$ is generated by pairs of points distance 1 apart with nothing in between them. That's *all* pairs of points distance 1 apart, and there are two such pairs for each integer (two, because they're *ordered* pairs).
- $\ell \geq 2$: a pair of points of $\mathbb{Z}$ distance 2 or more apart always has something strictly in between them, so $H_{1,2}(\mathbb{Z}) = H_{1,3}(\mathbb{Z}) = \cdots = 0$.

Actually, this metric space is a graph, so presumably everything I've just said follows from the **Hepworth–Willerton paper** [https://arxiv.org/abs/1505.04125] .

The point is that although the metric space $\mathbb{Z}$ fails to be Menger convex, it only fails because of the points distance 1 apart; for further-separated points it's fine. And the intuition that $\mathbb{Z}$ is "nearly but not quite Menger convex" is given precise expression by the fact that $H_{1,\ell}(\mathbb{Z}) = 0$ for all but one value of ℓ.

Posted by: Tom Leinster on September 6, 2016 4:55 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

First off, let me add my thanks for writing this post. The other thread long since outstripped both my actual ability to follow it in real time, and any ambition I might have had to go back and make sense of it later.

> ***1st magnitude homology measures lack of convexity.***

Awesome! This sounds like exactly the kind of geometric information we would hope to see encoded in a homology theory for metric spaces.

Just before seeing this comment I was wondering if negative type could be characterized in terms of magnitude homology. Any thoughts there?

Incidentally, the hypothesis that no two points have a point strictly between them, or equivalently that the triangle inequality is always strict, has come up a couple times. I can't remember precisely where, but I've seen a hypothesis similar to that somewhere before. (Maybe in Nik Weaver's book ***Lipschitz Algebras*** [https://books.google.com/books?isbn=9810238738] ?) Here's one way of producing many examples of spaces with this property: given any metric space (X, d) and $\alpha \in (0, 1)$, the triangle inequality is always strict in the metric space (X, d^α). It may be that the condition that I'm dimly remembering is actually that the metric is the α power of a metric.

Posted by: Mark Meckes on September 6, 2016 6:18 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Damn, I thought I dimly remembered that there was a *name* for metric spaces in which the triangle inequality is always strict, and that you (Mark) had once told me that name. I was trying to remember it a week or two ago, but it seems that if you ever knew it, you've forgotten too!

Posted by: Tom Leinster on September 6, 2016 7:54 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

The first homology of a metric space says something about the *existence* of geodesic paths between points. I haven't got a full description of second homology, but it seems that it has something to do with *uniqueness*/*multiplicity* of geodesics.

Specifically, I claim that $H_{2,\ell}(X) = 0$ for any convex subset X of $\mathbb{R}^n$ and any $\ell \geq 0$. And I think the reason for this is to do with the fact that in $\mathbb{R}^n$, there's a *unique* shortest path between any two points.

The calculation is below, but before getting stuck in, let me point out that matters are very different for graphs. I tend to think of graphs as the metric spaces that are as *unlike* subspaces of Euclidean space as it's possible to be. For instance, in any metric space we can ask how many midpoints exist between a given pair of points. (By a midpoint I mean a point whose distance to each of the two given points is half the overall distance.) In a subspace of $\mathbb{R}^n$, a given pair of points has at most one midpoint, but in a graph there can be any number of them.

Anyway, Richard and Simon computed lots of examples of magnitude homology of graphs in their paper, and H_2 very often isn't zero. For instance, the 5-cycle C_5 has

$$H_{2,2}(C_5) = \mathbb{Z}^{20}, \quad H_{2,3}(C_5) = \mathbb{Z}^{40}, \quad H_{2,4}(C_5) = \mathbb{Z}^{20}$$

(and the rest are zero).

OK. Now I'll prove my claim that when X is a convex subset of $\mathbb{R}^n$, the homology groups $H_{2,\ell}(X)$ are trivial for all $\ell \geq 0$.

A typical element of $\mathrm{MC}_{2,\ell}(X)$ is a linear combination

$$\alpha = \sum_{x,y,z} a_{xyz}(x, y, z)$$

where the sum is over all points $x \neq y \neq z$ such that $d(x, y) + d(y, z) = \ell$, the coefficients a_{xyz} are integers, and all but finitely many coefficients are zero. By the **formula for ∂ I just mentioned** [https://golem.ph.utexas.edu/category/2016/09/magnitude_homology.html#c051279],

$$\partial(\alpha) = -\sum_{x \neq z \,:\, d(x,z)=\ell} \left(\sum_{y \text{ strictly between } x \text{ and } z} a_{xyz} \right) (x, z).$$

So, α is a cycle if and only if for all x and z such that $d(x, z) = \ell$,

$$\sum_{y \text{ strictly between } x \text{ and } z} a_{xyz} = 0.$$

Suppose now that α *is* a cycle. To prove my claim, I have to show that it is a boundary.

The sum $\alpha = \sum a_{xyz}(x, y, z)$ splits into two parts: those for which y is between x and z, and those for which it's not. Those for which it's not are boundaries, since by convexity we can choose some u strictly between y and z, and then

$$\partial(x, y, u, z) = (x, y, z).$$

So we can assume that $a_{xyz} = 0$ unless x, y and z are collinear. Now, fixing x and z such that $d(x, z) = \ell$, it's enough to prove that

$$\alpha_{xz} := \sum_{y \text{ strictly between } x \text{ and } z} a_{xyz}(x, y, z)$$

is a boundary. But we know that

$$\sum_{y \text{ strictly between } x \text{ and } z} a_{xyz} = 0,$$

and from this it follows that α_{xz} can be expressed as a $\mathbb{Z}$-linear combination of expressions of the form

$$(x, y_1, z) - (x, y_2, z)$$

where y_1 and y_2 are both strictly between x and z.

Now we use something special about $\mathbb{R}^n$ — something closely related to the uniqueness of geodesics. Whenever we have points y_1 and y_2 both between x and z, *they must all lie on a line*. That is, one of the following two possibilities must occur:

- $d(x, y_1) + d(y_1, y_2) + d(y_2, z) = d(x, z)$, or
- $d(x, y_2) + d(y_2, y_1) + d(y_1, z) = d(x, z)$.

That's not true for a general metric space. For instance, consider the geodesic metric on the sphere. My home (y_1) is between the north pole (x) and the south pole (z), and your home (y_2) is too, but my home probably isn't between your home and the north pole or vice versa.

Back to the calculation. It remains to prove that when y_1 and y_2 are points strictly between x and z, the element

$$(x, y_1, z) - (x, y_2, z)$$

of $\mathrm{MC}_{2,\ell}(X)$ is a boundary. If $y_1 = y_2$ that's immediate. If not, we have four distinct points on a line, WLOG in the order x, y_1, y_2, z. And then $-(x, y_1, y_2, z) \in \mathrm{MC}_{3,\ell}(X)$, with

$$\partial(-(x, y_1, y_2, z)) = (x, y_1, z) - (x, y_2, z),$$

as required.

Posted by: Tom Leinster on September 6, 2016 8:57 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

That makes sense. The statement that (x, y, z) is a boundary if y is not between x and z is also using something special about $\mathbb{R}^n$, right? Something like that if y is between x and z, and z is between y and w, then y and z are between x and w?

Now I'm tempted to conjecture something about $H_{2,\ell}(S^1)$. Like maybe that it's zero unless $\ell = \pi$, in which case it's generated by something to do with pairs of antipodal points. But that could be way off…

Posted by: Mike Shulman on September 6, 2016 10:15 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Yes, I agree that this step also uses something special about $\mathbb{R}^n$. I only spotted that after posting.

I've run into a few similar "betweenness" properties of metric spaces. I think at least one of them might even have a name. But I'm not at all on top of them.

I was thinking that it would be good to compute $H_{2,\ell}(S^1)$ with its *geodesic* metric; is that what you had in mind? Oh, I guess it must be, because if it was the Euclidean metric then the differentials would all be zero.

Posted by: Tom Leinster on September 6, 2016 10:26 PM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Tom, thinking of the antipodal points on spheres case, did you miss out a word **above** [https://golem.ph.utexas.edu/category/2016/09/magnitude_homology.html#c051293] :

> In a subspace of $\mathbb{R}^n$, a given pair of points has at most one midpoint, but in a graph there can be any number of them?

Posted by: David Corfield on September 7, 2016 8:21 AM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Sorry for the noise, I'm thinking of the geodesic metric and you weren't.

Posted by: David Corfield on September 7, 2016 8:39 AM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Wow! I think this is our first indication that the magnitude homology of an infinite metric space carries interesting information.

What about H_2? (-:

Posted by: Mike Shulman on September 6, 2016 9:01 PM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

> What about H_2? (-:

I see that you're ahead of me! (4 minutes ahead of me, to be precise.)

Posted by: Mike Shulman on September 6, 2016 10:00 PM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

You were looking at the wrong table in our paper when you wrote

$$H_{2,2}(C_5) = \mathbb{Z}^{20}, \quad H_{2,3}(C_5) = \mathbb{Z}^{40}, \quad H_{2,4}(C_5) = \mathbb{Z}^{20}$$

They are actually the chain groups, not the homology groups. The actual non-trivial ones are

$$H_{2,2}(C_5) = \mathbb{Z}^{10}, \quad H_{2,3}(C_5) = \mathbb{Z}^{10}.$$

I believe that $H_{2,2}(G)$ for a graph G is trivial if and only if the graph is discrete, ie. has no edges.

On the one hand if the graph is discrete then $H_{0,0}(G)$ is the only non-trivial homology group.

On the other hand, Owen Biesel **told us** [https://golem.ph.utexas.edu/category/2013/04/tutte_polynomials_and_magnitud.html#c043766] that the coefficient of q^2 in graph magnitude is $2E + 6\Delta + 2\Lambda'$, whatever this is, but in particular it is at least $2E$, which is twice the number of edges of the graph.

The coefficient of q^2 is $\mathrm{rank}(H_{2,2}(G)) - \mathrm{rank}(H_{1,2}(G))$, thus $\mathrm{rank}(H_{2,2}(G)) \geq 2E$. So $H_{2,2}(G)$ is non-trivial if the graph has any edges.

Posted by: Simon Willerton on September 6, 2016 11:02 PM | Permalink | Reply to this

Re: Magnitude Homology
[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Nice! And thanks for the correction.

Posted by: Tom Leinster on September 7, 2016 5:51 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Backing all the way up to the Euler characteristic / magnitude of a finite category: does this construction shed any light on the meaning of weightings and coweightings (which don't appear in this post at all)?

Posted by: Mark Meckes on September 7, 2016 3:31 PM | Permalink | Reply to this

Re: Magnitude Homology

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Delurking to ask a non-mathematical question: is there any way to adjust the height of the vertical boxes that contain the LaTeX-like formulas? (I'm trying to print this post for later reading and the large vertical spacing is leading to a worrying number of pages.)

Posted by: Yemon Choi on September 7, 2016 5:33 PM | Permalink | Reply to this

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