An ABC of Category Theory

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Category theory sheds light on the following phenomena, facts, and concepts:

A finite-dimensional vector space is 'naturally' isomorphic to its double dual

An n-element set admits $n!$ total orderings and $n!$ permutations, but there is no 'natural' correspondence between the two

Affine varieties are 'dual' to certain rings

Linear algebra can be done equivalently with matrices or linear transformations

Simplicial sets

The geometric realization of a simplicial set

Classifying spaces

Fields of fractions

Free groups

Group rings

The abelianization of a group

Stone-Čech compactification

The universal enveloping algebra of a Lie algebra

The completion of a metric space

Sheaves, étale maps, and sheafification

The tensor product of modules

The tensor product of commutative rings

The wedge of based spaces

Germs

The p-adic integers

$f^{-1}(A \cup B) = f^{-1}A \cup f^{-1}B$,
$f^{-1}(A \cap B) = f^{-1}A \cap f^{-1}B$, and
$f(A \cup B) = fA \cup fB$, but
$f(A \cap B) \neq fA \cap fB$ in general

Van Kampen's theorem on the fundamental group of a union of spaces

The vector space dual of $k[X]$ (polynomials) is $k[\![X]\!]$ (formal power series)

Direct products of groups are easy, but free products are hard

Kernels of sheaf morphisms are easy, but cokernels are hard

Universal algebra

The inverse of a bijective homomorphism of groups or rings is a homomorphism, and similarly for continuous maps of compact Hausdorff spaces, but not for arbitrary spaces

Homology and cohomology

Ext and Tor

By the end of the course, for most of these, you should be able to see how.

Lecture 0

Informal introduction

This lecture is about the notion of universal property, which is fundamental to category theory and occupies a large part of this course.

Example 0.1 Everyone knows that the 'canonical' way to turn an integral domain into a field is to take its field of fractions. Here's how to make this precise.

Let R be an integral domain. Then its field of fractions $F(R)$ is the 'universal field containing R as a subring': given a field k and an embedding $f : R \longrightarrow k$, there is a unique embedding $\overline{f} : F(R) \longrightarrow k$ such that $\overline{f}(r/1) = f(r)$ for all $r \in R$. (Here 'embedding' means 'injective ring homomorphism'.) This is depicted as

$$\begin{tikzcd} R \arrow[r, "\text{inclusion}"] \arrow[dr, "\text{embedding } f"'] & F(R) \arrow[d, dotted, "\exists!\text{ embedding } \overline{f} \text{ of fields}"] \\ & k. \end{tikzcd}$$

Example 0.2 Let V be a vector space with a basis $(v_i)_{i \in I}$. Then for any vector space W, there is a 'natural' bijection

$$\text{linear maps } V \longrightarrow W \qquad \leftrightsquigarrow \qquad \text{functions } I \longrightarrow W.$$

To see this, define a function $b : I \longrightarrow V$ by $b(i) = v_i$. Then it is a simple exercise to prove the following universal property:

$$\begin{tikzcd} I \arrow[r, "b"] \arrow[dr, "f"'] & V \arrow[d, dotted, "\exists!\text{ linear } \overline{f}"] \\ & W. \end{tikzcd}$$

This means: for all vector spaces W and all functions $f : I \longrightarrow W$, there exists a unique linear map $\overline{f} : U \longrightarrow V$ such that $\overline{f}(v_i) = f(i)$ for all $i \in I$. We then have a correspondence $\overline{f} \leftrightsquigarrow f$, as claimed.

Example 0.3 Given vector spaces U and V, there is a 'universal bilinear map out of $U \times V$'. In other words, there are a certain vector space T and a certain bilinear map $b : U \times V \longrightarrow T$ with the following universal property:

$$\begin{tikzcd} U \times V \arrow[r, "\text{bilinear } b"] \arrow[dr, "\text{bilinear } f"'] & T \arrow[d, dotted, "\exists!\text{ linear } \overline{f}"] \\ & W. \end{tikzcd} \qquad (1)$$

(In fact, T is the tensor product $U \otimes V$, for those who know what that means.)

A universal property always determines the object in question up to isomorphism (and who could want more?). For instance, in Example 0.3 we have:

Proposition 0.4 *Let U and V be vector spaces. Suppose that $b : U \times V \longrightarrow T$ and $b' : U \times V \longrightarrow T'$ are both universal bilinear maps out of $U \times V$. Then $T \cong T'$.*

Proof Take $(U \times V \xrightarrow{f} W) = (U \times V \xrightarrow{b'} T')$ in (1) to obtain a linear map $i : T \longrightarrow T'$ satisfying $i \circ b = b'$. Similarly, use the universality of b' to obtain a linear map $i' : T' \longrightarrow T$ satisfying $i' \circ b' = b$.

We now have a linear map $i' \circ i : T \longrightarrow T$ with $(i' \circ i) \circ b = \mathrm{id}_T \circ b$, as illustrated:

$$\begin{tikzcd}
 & T \arrow[d, "i"] \\
U \times V \arrow[ur, "b"] \arrow[r, "b'"] \arrow[dr, "b"'] & T' \arrow[d, "i'"] \\
 & T
\end{tikzcd}$$

By the uniqueness part of (1), this implies that $i' \circ i = \mathrm{id}_T$. Similarly, $i \circ i' = \mathrm{id}_{T'}$. Hence $T \cong T'$. $\square$

The lesson is that once we've found a universal property of an object, we can ignore its explicit construction: the universal property tells us everything we need to know. For instance, there are several ways of constructing the tensor product of vector spaces, but once you've proved (1) above you can forget the construction entirely without losing any information. The same principle applies to the product of sets. Few mathematicians could write down the standard set-theoretic definition of the cartesian product of sets (or equivalently, of ordered pair), but every mathematician knows that in practice you don't *need* to know this definition. All you actually need to know about the product is its universal property:

Example 0.5 Let X_1 and X_2 be sets. Then the product-projections

$$\begin{array}{ccccc}
X_1 & \xleftarrow{\mathrm{pr}_1} & X_1 \times X_2 & \xrightarrow{\mathrm{pr}_2} & X_2 \\
x_1 & \longleftarrow\!\shortmid & (x_1, x_2) & \longmapsto & x_2
\end{array}$$

have the following universal property:

$$\begin{tikzcd}
 & W \arrow[dl, "f_1"'] \arrow[d, dotted, "\exists! f"] \arrow[dr, "f_2"] & \\
 & X_1 \times X_2 \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
X_1 & & X_2.
\end{tikzcd}$$

That is: for any set W and pair of functions $f_1 : W \longrightarrow X_1$, $f_2 : W \longrightarrow X_2$, there is a unique function $f : W \longrightarrow X_1 \times X_2$ such that $\mathrm{pr}_1 \circ f = f_1$ and $\mathrm{pr}_2 \circ f = f_2$. (Namely, $f(w) = (f_1(w), f_2(w))$.)

Example 0.6 The same universal property holds for the product of topological spaces X and Y, where this time f_1, f_2 and f are *continuous* maps. The universal property holds because the 'product topology' on $X \times Y$ is the *smallest* topology for which pr_1 and pr_2 are continuous.

Example 0.7 Let $f : A \longrightarrow B$ be a homomorphism of abelian groups; then there is a diagram

$$\begin{tikzcd}
\ker(f) \arrow[r, hook, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B.
\end{tikzcd}$$

The inclusion map $i : \ker(f) \hookrightarrow A$ satisfies $f \circ i = 0 \circ i$ and 'is universal with this property', in a sense to be supplied by the diligent reader.

Universal properties appear in various different guises, and accordingly it is convenient to have various different formalisms. Thus Examples 0.1 and 0.2 are most readily described as 'adjoint functors', Example 0.3 as a 'representable functor', and Examples 0.5–0.7 as 'limits'. (In principle, anything that can be described in one formalism can be described in any of the others, in the same way that anything done in cartesian coordinates can in principle be done in polar coordinates and *vice versa*.) Here are some examples of 'colimits', the dual concept of 'limits' in the sense that all the arrows get reversed.

Example 0.8 Let $f : B \longrightarrow A$ be a homomorphism of abelian groups. Then the quotient map

$$q : A \longrightarrow \mathrm{coker}(f) = B/\mathrm{im}(f)$$

has the universal property shown:

$$\begin{tikzcd}
\operatorname{coker}(f) \arrow[d, dotted, "\exists!\overline{g}"'] & A \arrow[l, "q"'] \arrow[dl, "g"] & B \arrow[l, shift right, "f"'] \arrow[l, shift left, "0"]\\
Z. & &
\end{tikzcd}$$

In other words, given an abelian group Z and a homomorphism $g : A \longrightarrow Z$ satisfying $g \circ f = 0$, there is a unique homomorphism $\overline{g} : \operatorname{coker}(f) \longrightarrow Z$ satisfying $\overline{g} \circ q = g$.

Example 0.9 Let X be a topological space and let $E \subseteq X \times X$ be an equivalence relation on X. Then the set X/E of equivalence classes can be given the quotient topology, and we have a diagram

$$\begin{tikzcd}
X/E & X \arrow[l, "q"'] & E \arrow[l, shift right, "\mathrm{pr}_1"'] \arrow[l, shift left, "\mathrm{pr}_2"]
\end{tikzcd}$$

where the pr_i are the projections and q is the quotient map. This diagram has a universal property precisely analogous to that in the previous example. The universal property holds because the quotient topology is *largest* such that q is continuous; compare Example 0.6.

Example 0.10 The circle S^1 also has a universal property of this kind, by regarding it as a real interval with the endpoints identified:

$$\begin{tikzcd}
S^1 & {[0,1]} \arrow[l] & \{\star\} \arrow[l, shift right, "0"'] \arrow[l, shift left, "1"]
\end{tikzcd}$$

Here $\{\star\}$ is a ('the'?) one-point space.

Example 0.11 For the final example, let X be a topological space covered by two open subsets: $X = U \cup V$. Then the diagram of inclusion maps

$$\begin{tikzcd}
U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "j'"] \\
V \arrow[r, "i'"'] & X
\end{tikzcd}$$

has a universal property in the world of topological spaces and continuous maps:

$$
\begin{tikzcd}
U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "j'"] \arrow[ddr, "f"] & \\
V \arrow[r, "i'"'] \arrow[drr, "g"'] & X \arrow[dr, dotted, "\exists! h"] & \\
& & Y.
\end{tikzcd}
$$

That is, given Y, f and g such that $f \circ i = g \circ j$, there is precisely one h satisfying $h \circ j' = f$ and $h \circ i' = g$.

Under favourable conditions, the induced diagram

$$
\begin{tikzcd}
\pi_1(U \cap V) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(U) \arrow[d, "j'_*"] \\
\pi_1(V) \arrow[r, "i'_*"'] & \pi_1(X)
\end{tikzcd}
$$

of fundamental groups has the same property in the world of groups and group homomorphisms. This is *van Kampen's Theorem.* In fact, van Kampen stated his theorem in a rather complicated way, in terms of generators and relations, as it was the 1930s and categories had not yet been invented. But using the language of colimits, the result can be stated transparently.

Exercises

0.12 Find three more examples of universal properties.

0.13 Choose one of your examples and prove that the universal property determines the object possessing it uniquely up to isomorphism. (In other words, prove the analogue of Proposition 0.4.)