Solutions to exercises from Lecture 1

1.18 Given spaces X and Y, homotopy defines an equivalence relation on the set $\mathbf{Top}(X,Y)$ of continuous maps from X to Y; write $[f]$ for the equivalence class of a continuous map f. It is meant to be implicit in the question that in the category **Toph**, the identity on a space X is $[1_X]$ and the composite of maps

$$X \xrightarrow{[f]} Y \xrightarrow{[g]} Z$$

is $[g \circ f]$. For the latter to make sense, we need to know that given a diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & Y \arrow[r, shift left, "g_0"] \arrow[r, shift right, "g_1"'] & Z \end{tikzcd}$$

of spaces and continuous maps, if f_0 is homotopic to f_1 and g_0 is homotopic to g_1 then $g_0 \circ f_0$ is homotopic to $g_1 \circ f_1$. (Proof strategy: if $(f_t)_{0 \leq t \leq 1}$ and $(g_t)_{0 \leq t \leq 1}$ are continuously parametrized families of maps then so is $(g_t \circ f_t)_{0 \leq t \leq 1}$.) The associativity and unit axioms for **Toph** follow immediately from those for **Top** (or **Set**).

Two spaces X and X' are isomorphic as objects of **Toph** if and only if there exist mutually inverse maps $\begin{tikzcd} X \arrow[r, shift left, "\phi"] & X' \arrow[l, shift left, "\phi'"] \end{tikzcd}$ in **Toph**. This says that there exist continuous maps $\begin{tikzcd} X \arrow[r, shift left, "f"] & X' \arrow[l, shift left, "f'"] \end{tikzcd}$ such that $[f'] \circ [f] = 1_X$ (where this '1_X' is the identity on X in **Toph**) and $[f] \circ [f'] = 1_{X'}$. Since $[f'] \circ [f] = [f' \circ f]$ and $1_X = [1_X]$, the first equation says that $f' \circ f$ is homotopic to the identity 1_X on X, and similarly the second. So isomorphism in **Toph** is just homotopy equivalence.

1.19 If monoids are regarded as one-object categories then homomorphisms between monoids are the same as functors between the corresponding categories. So two monoids are isomorphic if and only if their corresponding categories are isomorphic.

The opposite of a category with only one object still has only one object. So any monoid M gives rise to a new monoid M^{op}; it has the same underlying set and unit, and if the multiplication of M is written $\circ$ then the multiplication $\Box$ of M^{op} is given by $g \Box f = f \circ g$.

Let G be a group. By the comments above, we have to show that there is an isomorphism of groups $\phi : G \xrightarrow{\sim} G^{\mathrm{op}}$. Take $\phi(g) = g^{-1}$: then ϕ is a homomorphism since

$$\phi(g \circ f) = (g \circ f)^{-1} = f^{-1} \circ g^{-1} = g^{-1} \Box f^{-1} = \phi(g) \Box \phi(f),$$

and ϕ is clearly a bijection, so ϕ is an isomorphism.

Let M be the monoid of endomorphisms of a two-element set $\{x, y\}$, with composition as multiplication. (The idea of a group is that it's the automorphisms of some object; the idea of a monoid is that it's the endomorphisms of some object. So this is the 'simplest' non-trivial monoid.) Then M has $2^2 = 4$ elements: the identity, a transposition τ, and two constant functions Δx, Δy. We have $(\Delta x) \circ f = \Delta x$ for all $f \in M$. Suppose for a contradiction that $M \cong M^{\mathrm{op}}$. Then there is some $e \in M^{\mathrm{op}}$ such that $e \Box f = e$ for all $f \in M^{\mathrm{op}}$; that is, there is some $e \in M$ such that $f \circ e = e$ for all $f \in M$. In particular, $\tau(e(x)) = e(x)$, which is impossible.

1.20 Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and suppose that A and A' are isomorphic objects of $\mathcal{A}$. Then there is an isomorphism $f : A \longrightarrow A'$, so there are maps

$$FA \underset{F(f^{-1})}{\overset{Ff}{\rightleftarrows}} FA'$$

in $\mathcal{B}$. Indeed, Ff and $F(f^{-1})$ are mutually inverse, since

$$(F(f^{-1})) \circ (Ff) = F(f^{-1} \circ f) = F1_A = 1_{FA}$$

and dually. So $FA \cong FA'$.

(This proof shows that if $A \cong A'$ then $FA \cong FA'$, as requested, but also that if f is an isomorphism then so is Ff, and $(Ff)^{-1} = F(f^{-1})$.)