Rings graded equivalent to the first Weyl algebra

Susan J. Sierra
ssierra@umich.edu

University of Michigan

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Outline

1. Introduction
2. Picard groups and equivalences of graded categories
3. Rings graded equivalent to the Weyl algebra

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1 Introduction

2 Picard groups and equivalences of graded categories

3 Rings graded equivalent to the Weyl algebra

The basic question

Problem

Which graded rings have the same graded module category as the Weyl algebra?

The Euler gradation on the Weyl algebra:

- Let $A = \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$.
- Let $x \in A_1$, $y \in A_{-1}$.

Notations and definitions

Conventions:

- *Graded ring* means $\mathbb{Z}$-graded.
- All rings are $\mathbb{C}$-algebras.

Let R be a graded ring. The category gr-R is defined by:

- Objects are graded right R-modules: $M = \bigoplus_{i\in\mathbb{Z}} M_i$ with $M_i R_j \subseteq M_{i+j}$.
- Morphisms are maps of degree 0.
- Write:
  - $\hom_R(M,N)$ for $\mathrm{Hom}_{\text{gr-}R}(M,N)$;
  - $\mathrm{Hom}_R(M,N)$ for $\mathrm{Hom}_{\text{mod-}R}(M,N)$.
- So if $f \in \hom_R(M,N)$, we have $f(M_i) \subseteq N_i$ for all i.

If gr-$R \simeq$ gr-S, we say that R and S are *graded equivalent*.

Motivation

- The analogous question for the ungraded module category is very interesting: see results of Stafford, Cannings-Holland, and Berest-Wilson.
- A good test case for general results on equivalences of graded module categories.

Example 1

Proposition

A is graded equivalent to an idealizer in a localization of A.

- Let $T = A[y^{-1}]$.
- Let $B = \mathbb{C} + xT$.
- B is the *idealizer* in T of the right ideal xT: $B = \{\theta \in T \mid \theta xT \subseteq xT\} = \mathbb{I}_T(xT)$.
- Then A and B are graded equivalent.

This is surprising:

- B has a (unique) two-sided ideal and a 1-dimensional simple module.
- In fact, B is a standard example of a ring failing the *second layer condition* on prime ideals.

Example 2

Let $A(2) = \bigoplus_{i \in \mathbb{Z}} A(2)_i$ be the *2nd Veronese of A*; that is, $A(2)_i = A_{2i}$.

Proposition

A and $A(2)$ are graded equivalent.

NB: This is not a statement about projective geometry.

Graded Morita equivalence

Let P be a finitely generated graded projective A-module.

- Then trivially $S = \mathrm{End}_A(P)$ is a graded ring and $\mathrm{Hom}_A(P, __)$ restricts to an equivalence between gr-A and gr-S.
- We call such a functor a *graded Morita equivalence*.
- When we classify rings graded equivalent to the Weyl algebra, we will do so up to graded Morita equivalence.

Graded Morita equivalence

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- When we classify rings graded equivalent to the Weyl algebra, we will do so up to graded Morita equivalence.

The full classification

Theorem

Let n be a positive integer, and let f and g be monic polynomials in $\mathbb{C}[z]$. Let $S(f;n)$ be the ring generated over $\mathbb{C}$ by x, y, and z, with relations

$$xz - zx = nx \qquad yz - zy = -ny$$
$$xy = f(z) \qquad yx = f(z-n)$$

and let

$$R(f,g;n) = \mathbb{C} + gS(f;n) \subseteq S(f;n).$$

Then up to graded Morita equivalence the rings graded equivalent to A are precisely the rings $R(f,g;n)$, where $fg = \prod_{i=0}^{n-1}(z+i)$.

What does this mean?

The details of the definition of $R(f, g; n)$ aren't important. Here are some examples:

- $R(z, 1; 1) \cong A$.
- $R(1, z; 1) \cong B$.
- $R(z(z+1), 1; 2) \cong A(2)$.

Proof of theorem: Via understanding the autoequivalence group of gr-A.

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The Picard group

We use the terminology:

Definition

Let R be a graded ring. The *Picard group of* gr-R is the group of autoequivalences of gr-R modulo natural isomorphism. We denote it by Pic(gr-R).

- By Morita's Theorem, the autoequivalence group of mod-R is isomorphic to the group of invertible R-bimodules.
- If R and S are graded equivalent, then $\mathrm{Pic}(\text{gr-}R) \cong \mathrm{Pic}(\text{gr-}S)$.
- If $\Phi : \text{gr-}R \to \text{gr-}S$ is an equivalence of categories, we let $\Phi^* : \mathrm{Pic}(\text{gr-}S) \to \mathrm{Pic}(\text{gr-}R)$ be the induced isomorphism. We call it *pulling back* by Φ.

The shift functor

Example: the shift functor

- If M is a graded R-module, define a new module $M\langle 1\rangle = \bigoplus_{n\in\mathbb{Z}} M\langle 1\rangle_n$ by $M\langle 1\rangle_n = M_{n-1}$.
- The functor $M \mapsto M\langle 1\rangle$ is an autoequivalence of gr-R. We denote it by $\mathcal{S}_R$.

Morita theory for graded rings

Theorem

Let R, S be graded rings. Then R and S are graded equivalent if and only if there is an autoequivalence $\mathcal{F} \in \mathrm{Pic}(\mathrm{gr}\text{-}R)$ and a graded right R-module P such that:

- *P is finitely generated projective;*
- *$\mathcal{F}$ is P-generative;*
- *$S \cong \mathrm{End}_R^{\mathcal{F}}(P)$.*

Furthermore, if $\mathcal{F}$ and P are as above, then $\mathrm{Hom}_R^{\mathcal{F}}(P, __)$: $\mathrm{gr}\text{-}R \to \mathrm{gr}\text{-}S$ is an equivalence of categories whose action on Picard groups sends $\mathcal{F}$ to $\mathcal{S}_S$.

Generative autoequivalences

Definition

Let $\mathcal{F}$ be an autoequivalence of gr-R and let P be a finitely generated graded projective R-module. If the modules $\{\mathcal{F}^n P\}_{n\in\mathbb{Z}}$ generate gr-R, then we say that $\mathcal{F}$ is *P-generative* or more generally *generative*.

- If P is a graded module that is a progenerator for mod-R, then $\mathcal{S}_R$ is P-generative.

Twisted endomorphism rings

Proposition (Part I)

Let R be a graded ring, let $\mathcal{F}$ be an autoequivalence of gr-R*, and let P be a graded right R-module. Then there is a natural way to put a ring structure on*

$$\bigoplus_{n\in\mathbb{Z}} \hom_R(\mathcal{F}^n P, P).$$

We call this the *$\mathcal{F}$-twisted endomorphism ring of P* or $\mathrm{End}_R^{\mathcal{F}}(P)$. The ring structure is given by:

$$\hom_R(\mathcal{F}^i P, P) \otimes \hom_R(\mathcal{F}^j P, P) \xrightarrow{1\otimes \mathcal{F}^i}$$

$$\hom_R(\mathcal{F}^i P, P) \otimes \hom_R(\mathcal{F}^{i+j} P, \mathcal{F}^i P) \longrightarrow \hom_R(\mathcal{F}^{i+j} P, P).$$

Twisted Hom functors

Proposition (Part II)

Let $S = \mathrm{End}_R^{\mathcal{F}}(P)$. *Then for any graded right* R*-module* N*, the object* $\bigoplus_{n\in\mathbb{Z}} \mathrm{hom}_R(\mathcal{F}^n P, N)$ *is naturally a graded right* S*-module, and*

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{hom}_R(\mathcal{F}^n P, __)$$

is a covariant functor from gr-R *to* gr-S.

We denote this functor by $\mathrm{Hom}_R^{\mathcal{F}}(P, __)$.

- Let $\mathcal{F} = \mathcal{S}_R$. Since for any finitely generated M, N, we have $\mathrm{Hom}_R(M, N) \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{hom}_R(M\langle n\rangle, N) = \mathrm{Hom}_R^{\mathcal{F}}(M, N)$, we have $\mathrm{Hom}_R^{\mathcal{F}}(M, __) \cong \mathrm{Hom}_R(M, __)$ and $\mathrm{End}_R^{\mathcal{F}}(M) \cong \mathrm{End}_R(M)$.

The Picard group and graded equivalent rings

Corollary

An autoequivalence $\mathcal{F} \in \mathrm{Pic}(\text{gr-}R)$ *is generative if and only if there is a ring* S *and an equivalence of categories* $\Phi : \text{gr-}R \to \text{gr-}S$ *such that* $\Phi^* \mathcal{S}_S \cong \mathcal{F}$.

Theorem

An equivalence of categories $\Phi : \text{gr-}R \to \text{gr-}S$ *is a graded Morita equivalence if and only if* $\Phi^* \mathcal{S}_S \cong \mathcal{S}_R$.

- NB: This result does *not* hold for arbitrary grading groups.
- We remark that if Φ is a so-called *Zhang twist* then the pullback of $\mathcal{S}_S$ is also of a special form.

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A useful corollary

Corollary

Let R, S, and S' be graded rings, and let Φ : gr-$R \to$ gr-S and Φ' : gr-$R \to$ gr-S' be equivalences of categories. Let $\mathcal{F} = \Phi^* \mathcal{S}_S$ and let $\mathcal{F}' = (\Phi')^* \mathcal{S}_{S'}$. Then S is graded Morita equivalent to S' if and only if $\mathcal{F}$ is conjugate to $\mathcal{F}'$ in Pic(gr-R).

Thus we have reduced the problem of classifying rings graded equivalent to A to the problem of understanding conjugacy classes in Pic(gr-A).

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The Veronese example

Recall that A and $A(2)$ are graded equivalent.
Proof:

- It's not hard to see that if $\mathcal{F} = \mathcal{S}_A^2$, then $A(2) \cong \mathrm{End}_A^{\mathcal{F}}(A)$.
- $A\langle 1\rangle$ and $A\langle -1\rangle$ generate A, since $xA + yA = A$.
- So $\{A\langle 2i\rangle\}$ generates gr-A and $\mathcal{F}$ is A-generative.
- So A and $A(2)$ are graded equivalent.

Simple objects in gr-A

The simple objects in gr-A are:

- A/xA and shifts.
- A/yA and shifts.
- $A/(xy+\lambda)A$ for $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.

Notation:

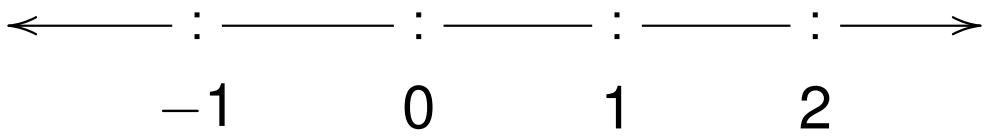
- Let $z = xy$, so $A_0 = \mathbb{C}[z]$.
- Let $X = A/xA$; let $Y = A/yA\langle 1\rangle$.
- There is an exact sequence:

$$0 \to Y \to A/zA \to X \to 0.$$

- So as a $\mathbb{C}[z]$-module, both X and Y are supported at (z). Likewise, $X\langle 1\rangle$ and $Y\langle 1\rangle$ are supported at $(z+1)$.

Involutions in Pic(gr-A)

A picture of gr-A:



Proposition

There is an autoequivalence ι_0 of gr-A *that switches X and Y and fixes all other simples; further, $\iota_0^2 \cong \mathrm{Id}_{\text{gr-}A}$.*

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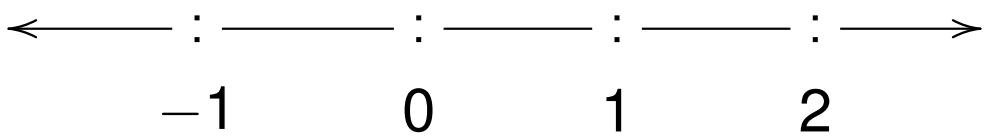

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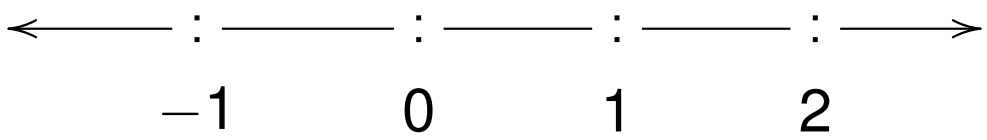

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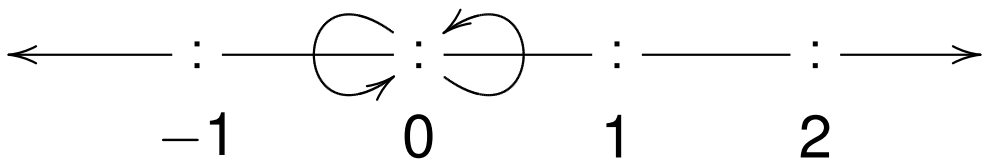


What is ι_0?

Constructing ι_0:

- Let P be a rank 1 projective
- Let $\iota_0(P)$ be the *reject* in P of $X \oplus Y$; i.e. the smallest submodule of P such that $P/\iota_0(P)$ is cogenerated by $X \oplus Y$ in gr-A.
- E.g. $\iota_0(A) = xA$ since $A/xA = X$.
- And $\iota_0(xA) = zA \cong A$.

Constructing B

Recall that we have $T = A[y^{-1}]$ and $B = \mathbb{C} + xT$.

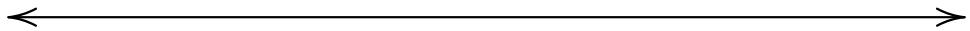
Proposition

Let $\mathcal{F} = \mathcal{S}_A \circ \iota_0$. Then $\mathcal{F}$ is A-generative and $B \cong \mathrm{End}_A^{\mathcal{F}}(A)$. Therefore B is graded equivalent to A.

Proof: By computing $\mathrm{End}_A^{\mathcal{F}}(A)$.

The category gr-B

Remark: $T = A[y^{-1}]$ is strongly graded, and $T_0 = A_0 = \mathbb{C}[z]$. So gr-$T \simeq$ mod-T_0; that is, (simple objects) in gr-T are parametrized by the complex line:



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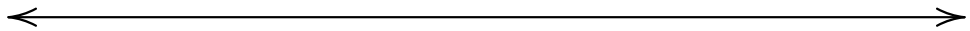

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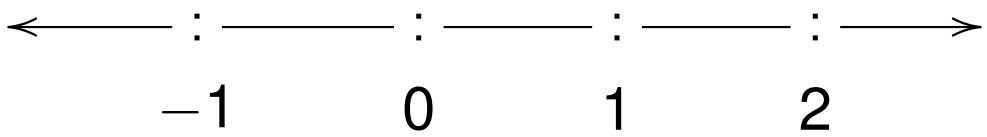

Why does this work?

So gr-B looks like gr-A. Why do we have a functorial relationship between the two?

- Combinatorics gives us many properties of B.
  - Note that $A\langle 1\rangle \cong \iota_0 A$ and so $\mathcal{F}(A\langle 1\rangle) \cong \mathcal{S}_A \iota_0^2 A \cong A\langle 1\rangle$.
  - Let $S = \mathrm{End}_A^{\mathcal{F}}(A)$. Then S has a maximal right ideal I that is invariant under shifting, and in fact is a two-sided ideal.
  - Further, $\mathcal{F}^n A \cong \iota_0 \iota_n A$ for all n, so $\hom_A(\mathcal{F}^n A, X) = 0$ if $n \neq 0$.
  - Let $\Phi = \mathrm{Hom}_A^{\mathcal{F}}(A, __)$. Thus $\Phi(X) \cong S/I \cong \mathbb{C}$.
- And in fact $S \cong B$.

What is Pic(gr-A)?

For any $j \in \mathbb{Z}$ have an involution ι_j; more generally have ι_J for any finite $J \subset \mathbb{Z}$.

Theorem

- $\mathrm{Pic}(\text{gr-}A) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}} \rtimes (\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}) = \{\iota_J\} \rtimes (\langle \mathcal{S}_A \rangle \rtimes \langle \omega \rangle)$.
- $\mathcal{F} \in \mathrm{Pic}(\text{gr-}A)$ *is generative if and only if* $\mathcal{F} \cong \mathcal{S}_A^n \circ \iota_J$, *with* $n \neq 0$.
- *If* $\mathcal{F}$ *is generative, then* $\mathcal{F}$ *is conjugate to a unique autoequivalence of the form* $\mathcal{S}_A^n \circ \iota_J$, *where* $n \geq 1$ *and* $J \subseteq \{0, \ldots, n-1\}$.

Relating the two results

Corollary

The graded Morita equivalence classes of rings graded equivalent to A are parametrized by the autoequivalences $\mathcal{S}_A^n \circ \iota_J$, *with* $n \geq 1$ *and* $J \subseteq \{0, \ldots, n-1\}$.

If $\mathcal{F} = \mathcal{S}_A^n \circ \iota_J$, then $\mathrm{End}_A^{\mathcal{F}}(A) \cong R(f, g; n)$ where J gives the roots of g; that is,

$$g = \prod_{j \in J} (z + j) \qquad f = \Big(\prod_{j=0}^{n-1} (z + i)\Big)/g.$$

The classification theorem, revisited

Theorem

Let n be a positive integer, and let f and g be monic polynomials in $\mathbb{C}[z]$. Let $S(f;n)$ be the ring generated over $\mathbb{C}$ by x, y, and z, with relations

$$xz - zx = nx \qquad yz - zy = -ny$$
$$xy = f(z) \qquad yx = f(z-n)$$

and let

$$R(f,g;n) = \mathbb{C} + gS(f;n) \subseteq S(f;n).$$

Then up to graded Morita equivalence the rings graded equivalent to A are precisely the rings $R(f,g;n)$, where $fg = \prod_{i=0}^{n-1}(z+i)$.

Final comments

- If $\mathcal{F} = \mathcal{S}_A^2 \circ \iota_1$, then $\mathrm{End}_A^{\mathcal{F}}(A) \cong R(z, z+1; 2)$. This is isomorphic to $A + (xy + 1/2)A = \mathbb{I}_A((xy + 1/2)A)$. So A is graded equivalent to an idealizer in A.
- One can also carry out a similar classification for other rings, such as primitive factors of $U(sl_2)$.

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