Pure and Applied Analysis – Tutorial Problems. The problems are of varying difficulty. The E questions are fairly routine exercises in the coursework, S questions are of standard difficulty and H questions may be quite challenging. P questions are more peripheral to the course.

Please hand in a solution to problem 1 in week 2 and a solution to problem 14 in week 3.

S1. In each of the following cases a bounded subset A of $\mathbb{R}$ is given. Find $\max A$ and $\min A$ (if they exist) as well as $\sup A$ and $\inf A$. Prove your claims.
(i) $A = \{(-1)^n : n \in \mathbb{N}\}$
(ii) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$
(iii) $A = \{x \in \mathbb{R} : x$ is rational and $0 \le x < 1\}$.

Solution. (i) $A = \{1, -1\}$ and then clearly $\max A = 1$ and $\min A = -1$. Hence $\sup A = 1$ and $\inf A = -1$.

(ii) We have $1 \in A$ and $\frac{1}{n} \le 1$ for all n so $\max A = 1$ and hence $\sup A = 1$.

Also 0 is a lower bound for A, and is the greatest lower bound since if $c > 0$ then we can find $n \in \mathbb{N}$ with $n > 1/c$, which implies $\frac{1}{n} < c$ so c is not a lower bound. So $\inf A = 0$. Since $0 \notin A$, $\min A$ does not exist.

(iii) We have $0 \in A$ and 0 is a lower bound for A so $0 = \min A$. 1 is an upper bound and if $c < 1$ then by the 'density of rationals' there is a rational x with $c < x < 1$,and then $c \in A$, so c is not an upper bound. So 1 is the least upper bound, i.e. $\sup A = 1$. As $1 \notin A$, $\max A$ does not exist.

S2. Let A be a non-empty bounded subset of $\mathbb{R}$. Define

$$D = \{|x - y| : x, y \in A\}$$

Show that D is bounded and that $\sup D = \sup A - \inf A$.

Solution. Let $M = \sup A$ and $m = \inf A$. Then for $x, y \in A$ we have $x \le M$ and $y \ge m$ so $x - y \le M - m$ and the same for $y - x$ hence $|x - y| \le M - m$ so D is bounded and $\sup D \le M - m$. To prove the reverse inequality let $\epsilon > 0$. Then we can find $x \in A$ with $x > M - \epsilon$ and $y \in A$ with $y < m + \epsilon$. Then $|x - y| \ge x - y > M - m - 2\epsilon$, so $\sup D > M - m - 2\epsilon$, for all $\epsilon > 0$, so $\sup D \ge M - m$ as required.

S3. Let A be a non-empty bounded subset of $\mathbb{R}$ and $\lambda \in (0, \infty)$. Define $\lambda A = \{\lambda a : a \in A\}$. Show that

$$\sup(\lambda A) = \lambda \sup A$$

What happens if $\lambda < 0$?

Solution. If $x \in \lambda A$ then for some $a \in A$, $x = \lambda a \le \lambda \sup a$. Hence $\sup(\lambda A) \le \lambda \sup A$. Conversely if $A \in A$ then $\lambda a \in \lambda A$ so $\lambda a \le \sup(\lambda A)$ and so $a \le \lambda^{-1} \sup(\lambda A)$. Thus $\sup A \le \lambda^{-1} \sup(\lambda A)$ and so $\sup(\lambda A) \ge \lambda \sup A$ as required.

If $\lambda < 0$ then in the same way we get $\sup(\lambda A) = \lambda \inf A$.

S4. Let A and B be two non-empty subsets of $\mathbb{R}$ such that for all $a \in A$ and all $b \in B$ we have $a \le b$.
(i) Show that $\sup A \le \inf B$.
(ii) Give an example of two non-empty sets A and B as above such that $\sup A = \sup B$ and $A \cap B = \varnothing$.
(iii) Suppose that for every $\epsilon > 0$ there exist $a \in A$ and $b \in B$ such that $|a - b| < \epsilon$. Show that $\sup A = \inf B$.

Solution. (i) If $b \in B$ then b is an upper bound for A so $\sup A \le b$. Hence $\sup A$ is a lower bound

for B and so $\sup A \le \inf B$.
(ii) $A = [0, 1)$ and $B = \{1\}$ will do.
(iii) If $\epsilon > 0$ and a, b are as in (iii) then $\inf B - \sup A \le b - a = |a - b| < \epsilon$. This holds for all $\epsilon > 0$ so $\inf B - \sup A \le 0$ and the result then follows from (i).

E5. Show that every non-empty open interval contains at least one irrational number.

Solution. Let the interval be (a, b) where $a < b$. Choose an integer $n > 2^{-1/2}(b-a)^{-1}$, and let k be the largest integer $\le 2^{-1/2}an$. Then $k+1 > 2^{-1/2}an$ so $2^{1/2}(k+1)/n > a$. On the other hand $2^{1/2}(k+1)/n = 2^{1/2}k/n + 2^{1/2}/n < a + (b-a) = b$. Thus $2^{1/2}(k+1)/n$ is an irrational number (since $2^{1/2}$ is irrational) in (a, b).

P6. Show that $\sqrt{2}$ is irrational.

Solution. Suppose $\sqrt{2} = p/q$ where p, q are relatively prime integers. Then $p^2 = 2q^2$ so p is even, but then p^2 is divisible by 4 so q must be even, a contradiction.

E7. Is $0.999\ldots$ strictly less than 1?

Solution. No - if x is the number with this decimal expansion, then for each n, we have $0.99\ldots 9 < x \le 1$, where there are n 9's, which means $x > 1 - 10^{-n}$ for every n, and it follows that $x = 1$.

E8. Is it true that every real number has a unique decimal expansion?

Solution. No - see previous problem.

S9. Let (a_n) be a sequence of real numbers and $a \in \mathbb{R}$. Show that if $a_n \to a$ then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \to a$$

Prove that the same is true if $a = \pm\infty$.

Solution. Write $b_n = n^{-1}(a_1 + a_2 + \cdots + a_n)$. The sequence (a_n) is bounded, so we can find M such that $|a_n| \le M$ for all n. Now let $\epsilon > 0$. Then there is $m > 0$ such for $n > m$ we have $|a_n - a| < \epsilon/2$. So for $n > m$, $|b_n - a| = \frac{1}{n}|(a_1 - a) + \cdots + (a_n - a)| \le \frac{2Mm}{n} + \epsilon/2$ where we have used $|a_k - a| \le 2M$ for $k \le m$. Then if also $n > 4Mm/\epsilon$ we get $|b_n - a| < \epsilon$, as required.

The case of an infinite limit is handled in the same way.

S10. In each of the following cases examine whether the given sequence converges. If it does, find its limit. If it doesn't find all its subsequential limits.
(a) $\frac{\log n}{n^\epsilon}$, where $\epsilon > 0$, **(b)** $\sqrt[n]{n}$, **(c)** $\sqrt[n]{n!}$, **(d)** $\frac{n!}{n^n}$, **(e)** $\frac{100^n}{n!}$, **(f)** $\sin\left(\frac{n\pi}{2}\right)$,
(g) $\sqrt{n^2+1} - n$, **(h)** $\sqrt{n^2+n} - n$.

Solution. (a) Writing $x = \log n$, $a_n = x/e^{\epsilon x} = x/(1 + \epsilon x + \epsilon^2 x^2/2 + \cdots) < x/(\epsilon^2 x^2/2) = 2\epsilon^{-2}x^{-1}$. As $n \to \infty$, $x \to \infty$ so $a_n \to 0$.
(b) $\log a_n = n^{-1}\log n \to 0$ by (a) so $a_n \to 1$.
(c) $\log a_n = \frac{1}{n}(\log 2 + \cdots + \log n) \to \infty$ by problem 9, since $\log n \to \infty$, so $a_n \to \infty$.
(d) $a_n = \frac{1}{n} \times \frac{2}{n} \times \cdots \times \frac{n}{n} < \frac{1}{n}$ so $a_n \to 0$.
(e) $\frac{a_{n+1}}{a_n} = \frac{100}{n+1} < \frac{1}{2}$ if $n > 200$ so $a_n \to 0$.
(f) a_n takes the values 0,1,0,-1 if $n = 4k, 4k+1, 4k+2, 4k+3$ respectively, so a_n does not converge but has subsequential limits 0, 1 and -1.
(g) $a_n = \frac{1}{n+\sqrt{n^2+1}} < n^{-1}$ so $a_n \to 0$.
(h) $a_n = \frac{n}{n+\sqrt{n^2+n}} = \frac{1}{1+\sqrt{1+n^{-1}}} \to \frac{1}{2}$.

S11. Show that $\left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$.

Solution. Writing $x = \frac{1}{n}$ we have $\log a_n = n \log(1 + \frac{1}{n}) = \frac{\log(1+x)}{x}$. Now as $n \to \infty$, $x \to 0$, and as $x \to 0$ we have $\frac{\log(1+x)}{x} \to 1$ by l'Hôpital's rule. So $\log a_n \to 1$ and hence $a_n \to e$.

E12. What is the limit of $\left(1 + \frac{1}{n^2}\right)^n$?

Solution. Applying l'Hôpital's rule in the same way as in problem 12 we find $\log a_n \to 0$ so $a_n \to 1$.

E13. A sequence a_n has the property, $\forall n \in \mathbb{N}, \quad a_{n+1} = 2a_n - 3$.
Show that
(i) If $a_1 = 3$ then for all $n \in \mathbb{N}$, $a_n = 3$.
(ii) If $a_1 > 3$ then $a_n \to +\infty$.
(iii) If $a_1 < 3$ then $a_n \to -\infty$

Solution. We have $a_{n+1} - 3 = 2(a_n - 3)$ from which the results follow.

S14. (i) Let (a_n) be a convergent sequence of real numbers and let A be the set of all its terms, i.e.

$$A = \{a_1, a_2, \ldots, a_n, \ldots\}$$

Prove that: A has a minimum or A has a maximum (or both).
(ii) Give an example of a bounded sequence (a_n) for which A has neither a minimum nor a maximum.

Solution. (i) Suppose $a_n \to a$. Since the sequence converges, A is bounded: let $s = \sup A$ and $l = \inf A$. First suppose $s > a$, and choose c with $s > c > a$. Then we can find m such that for $n > m$ we have $a_n < c$. Then we must have $A_n = s$ for some $n \leq m$, otherwise the supremum would be $< s$. So A has a maximum in this case. Similarly if $l < a$ then A has a minimum. The only other possibility is $s = l = a$ but in that case the sequence must be constant and $A = \{a\}$ so a is a max (and min).
(ii) $a_n = (-1)^n(1 - \frac{1}{n})$ is an example. The sup is 1 and the inf is -1, but neither is attained.

S15. Let (a_n) be a sequence of real numbers and $l \in \mathbb{R}$. Suppose that $a_{2n} \to l$ and $a_{2n-1} \to l$. Show that $a_n \to l$.

Solution. Let $\epsilon > 0$. Then we can find m so that $n > m$ implies $|a_{2n} - l| < \epsilon$ and $a_{2n-1} - l| < \epsilon$. Then if $k > 2m$, k must be either $2n - 1$ or $2n$ for some $n > m$, and in either case $|a_k - l| < \epsilon$ as required.

E16. Let (a_n) and (b_n) be two sequences of real numbers such that $a_n \to 0$ and (b_n) is bounded. Show that $a_n b_n \to 0$.

Solution. We have M such that $|b_n| \leq M$ for all n. Let $\epsilon > 0$. Then we find m such that $n > m$ implies $|a_n| < \epsilon/M$. Then for $n > m$ we have $|a_n b_n| \leq M|a_n| < M\epsilon/M = \epsilon$ as required.

E17. Find $\limsup a_n$ and $\liminf a_n$ in each of the following cases:
(i) $a_n = \frac{n+1}{n} + \cos \frac{n\pi}{2}$; (ii) $a_n = (-1)^n(\frac{n-1}{2n})$.

Solution. (i) We have $a_n \leq \frac{n+1}{n} + 1 = 2 + \frac{1}{n} \to 2$ so any subsequential limit must be ≤ 2. And $a_{4k} = 2 + \frac{1}{4k} \to 2$ so 2 is the largest subsequential limit so $\limsup a_n = 2$. Likewise $A_n \geq \frac{1}{n} \to 0$ and $a_{4k+2} = \frac{1}{4k+2} \to 0$ so 0 is the smallest subsequential limit, i.e. $\liminf a_n = 0$.
(ii) $a_n \leq \frac{n-1}{2n} \to \frac{1}{2}$ and $a_{2k} = \frac{2k-1}{2k} \to \frac{1}{2}$ so $\frac{1}{2}$ is the greatest subsequential limit so $\limsup a_n = \frac{1}{2}$. Similarly $\liminf a_n = -\frac{1}{2}$.

E18. Show that if (a_n) and (b_n) are bounded sequences then $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

Solution. Let $u_n = \sup(A_n, a_{n+1}, \cdots)$, $v_n = \sup(b_n, b_{n+1}, \cdots)$, $w_n = \sup(a_n + b_n, a_{n+1} + b_{n+1}, \cdots)$.

Then for all $m \geq n$, $a_m + b_m \leq u_n + v_n$ so $w_n \leq u_n + v_n$. Then $\limsup(a_n + b_n) = \lim w_n \leq \lim u_n + \lim v_n = \limsup a_n + \limsup b_n$.

H19. If a bounded sequence (a_n) of real numbers has only one subsequential limit $l \in \mathbb{R}$, show that it actually converges to l. [Hint: show that $\limsup a_n = \liminf a_n$]

Solution. $\limsup a_n$ and $\liminf a_n$ are both subsequential limits, hence both equal to l. Hence $\limsup a_n = \liminf a_n = l$ which implies $a_n \to l$.

E20. A sequence (a_n) of real numbers has the following property: There is an $l \in \mathbb{R}$ such that every subsequence of (a_n) has a subsequence converging to l. Show that $a_n \to l$. (Use the result of the last problem).

Solution. This follows from problem 17, since (a_n) cannot have any subsequence converging to any other limit (as that subsequence could not then have a subsequence converging to l.)

E21. Assuming $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.
[Answer: $\frac{\pi^2}{8}$]

Solution. $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}$ and subtracting this gives that the sum of the odd terms is $\frac{\pi^2}{8}$.

E22. Which one of the following sums is larger, $\sum_{n \text{ even}} \frac{1}{n^{10}}$ or $\sum_{n \text{ odd}} \frac{1}{n^{10}}$?

Solution. We have $\frac{1}{(2k-1)^{10}} > \frac{1}{(2k)^{10}}$ so pairing terms shows that the sum of the 'odd' series is larger.

E23. Let $\sum_n a_n$ be a convergent series of real or complex numbers. Show that its 'tail' converges to zero, i.e.

$$\sum_{n=N}^{\infty} a_n \to 0 \quad \text{as} \quad N \to \infty$$

Solution. Let S_n be the sum of the first n terms and S the sum, so $S_n \to S$. Then $\sum_{n=N}^{\infty} a_n = S - S_{N-1} \to 0$ since $S_{N-1} \to S$ as $N \to \infty$.

S24.. In each of the following cases examine whether the given series converges.

(a) $\displaystyle\sum_n \frac{100n+1}{4^n}$, **(b)** $\sum_{n=1}^{\infty} \frac{n^{\alpha}}{b^n}$ where $\alpha \in \mathbb{R}$ and $b > 1$, **(c)** $\displaystyle\sum_n \frac{3^n}{n!}$, **(d)** $\displaystyle\sum_n \frac{\cos n}{n^2}$,

(e) $\displaystyle\sum_n \frac{1}{\log n}$, **(f)** $\displaystyle\sum_n \frac{(-1)^n}{\sqrt{n}}$, **(g)** $\displaystyle\sum_n \left(\sqrt{n} - \sqrt{n-1}\right)$, **(h)** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ \log n\ \log\log n}$, **(i)** $\displaystyle\sum_n \sin\frac{n\pi}{2}$.

Solution. (a) Ratio test: $\frac{a_{n+1}}{a_n} = \frac{1}{4}(1 + \frac{1.01}{n}) \to \frac{1}{4}$ so series converges.
(b) Ratio test: $\frac{a_{n+1}}{a_n} = b^{-1}(1 + \frac{1}{n})^{\alpha} \to b^{-1} < 1$ so series converges.
(c) Ratio test: $\frac{a_{n+1}}{a_n} = \frac{3}{n+1} \to 0$ so series converges.
(d) $|a_n| \leq n^{-2}$ so series converges by comparison test
(e) $\frac{1}{\log n} > \frac{1}{n}$ so series diverges by comparison with $\sum \frac{1}{n}$.
(f) $n^{-1/2}$ is decreasing and converging to 0, so series converges by alternating series test.
(g) $a_n = (n^{1/2} + (n-1)^{1/2})^{-1} \geq \frac{1}{2}n^{-1/2}$ so series diverges by comparison with $\sum n^{-1/2}$.
(h) $a_n \geq \int_n^{n+1}\{x \log x \log\log x\}^{-1}dx$ so $\sum_{n=3}^{N} a_n \geq \int_3^{N+1}\{x \log x \log\log x\}^{-1}dx = [\log\log\log x]_3^{N+1} \to \infty$ as $N \to \infty$ so the series diverges.
(i) The terms do not tend to 0 (as $a_n = \pm 1$ when n is odd) so the series cannot converge.

You are encouraged to hand in a solution to problem 31 in week 4, 46 in week 5 and 50 in week 6.

S25. Let $\alpha, \beta \in (0, \infty)$ and consider the series $\sum_{n=2}^{\infty} \frac{1}{n^\alpha (\log n)^\beta}$. Show that:
(i) If $\alpha > 1$ then the series converges.
(ii) If $\alpha = 1$ and $\beta > 1$ then the series converges.
(iii) In all other cases the series diverges.

Solution. (i) For $n \geq 3$, $\log n > 1$ so $a_n < n^{-\alpha}$ and the series converges by comparison with $\sum n^{-\alpha}$.
(ii) Use integral test with $f(x) = x^{-1}(\log x)^{-\beta}$ which is decreasing to 0. We have $\int_2^N f(x)dx = (1-\beta)^{-1}[(\log n)^{1-\beta}]_2^N$ which converges to a finite limit as $N \to \infty$ so the series converges.
(iii) If $\alpha = \beta = 1$ then we apply the integral test with $f(x) = (x \log x)^{-1}$. We have $\int_2^N f(x)dx = [\log \log x]_2^N$ which $\to \infty$ as $n \to \infty$ so the series diverges. It hen follows from this by comparison that we have divergence when $\alpha = 1$ and $\beta < 1$ (or use integral test again).

Finally suppose $\alpha < 1$. Then it follows from problem 10(a) that $n^{1-\alpha}(\log n)^{-\beta} \to \infty$ as $n \to \infty$ and hence $a_n > n^{-1}$ for n large enough, so by comparison with $\sum n^{-1}$ the series diverges.

E26. Let $\sum_n a_n$ be a convergent series of positive terms. Show that $\sum_n a_n^2$ converges.

Solution. (a_n) must be bounded, i.e. for some $M > 0$ we have $a_n \leq M$ for all n. Then $a_n^2 \leq M a_n$ so by comparison $\sum a_n^2$ converges.

S27. Let (a_n) be a sequence of real numbers. If the series $\sum_n a_n^2$ converges, show that $\sum_n \frac{a_n}{n}$ converges. Is the converse true? [Hint: Use the Cauchy-Schwarz inequality.]

Solution. For any N we have $(\sum_{n=1}^N \frac{a_n}{n})^2 \leq \sum_{n=1}^N a_n^2 \sum_{n=1}^N n^{-2}$ by Cauchy-Schwarz, and as both series on the right have bounded partial sums so does $\sum \frac{a_n}{n}$, which therefore converges.

The converse is false. For example, let $a_n = n^{-1/2}$. Then $\sum \frac{a_n}{n} = \sum n^{-3/2}$ converges but $\sum a_n^2 = \sum n^{-1}$ does not.

E28. Let (a_n) be a decreasing sequence converging to zero. Show that for every $N \in \mathbb{N}$ we have
(i) $a_{2N} - a_{2N+1} + \cdots \geq 0$.
(ii) $-a_{2N+1} + a_{2N+2} - \cdots \leq 0$

Solution. (i) follows from $(a_{2N} - a_{2N+1}) \geq 0$, $(a_{2N+2} - a_{2N+3}) \geq 0$ etc, and (ii) is similar.

P29. If (a_n) is a sequence of positive real numbers numbers with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \in \mathbb{R}$ then $\lim_{n\to\infty} \sqrt[n]{a_n} = l$.

Solution. Let $c = \log l$ and define $b_1 = \log a_1$ and $b_n = \log a_n - \log a_{n-1}$ for $n > 1$. Then $b_n = \log \frac{a_n}{a_{n-1}} \to c$, so by problem 9, $\frac{1}{n}(b_1 + \cdots + b_n) \to c$, which means $\frac{1}{n} \log a_n \to c$ and hence $a_n^{1/n} \to l$.

S30. Define a sequence (a_n) by $a_{2n} = 1$, $n = 1, 2, 3, \ldots$ and $a_{2n+1} = 2^{\sqrt{n}}$, $n = 0, 1, 2, 3, \cdots$. Show that $\sqrt[n]{a_n} \to 1$ but $\frac{a_{n+1}}{a_n}$ has no limit.

Solution. $a_{2n}^{1/(2n)} = 1$ and $a_{2n+1}^{1/(2n+1)} = 2^{n^{1/2}/(2n+1)} \to 2^0 = 1$ so $a_n^{1/n} \to 1$. But $\frac{a_{2n+1}}{a_{2n}} = 2^{n^{1/2}} \to \infty$ while $\frac{a_{2n}}{a_{2n-1}} = 2^{-(n-1)^{1/2}} \to 0$ so $\frac{a_{n+1}}{a_n}$ has no limit.

S31. Let (a_n) be a sequence of real numbers such that $a_n \to 0$ as $n \to \infty$. Prove that there exists a subsequence (a_{n_k}) such that $\sum_k a_{n_k}$ converges. [Hint: Construct a subsequence (a_{n_k}) such that for all k, $|a_{n_k}| \leq \frac{1}{k^2}$.]

Solution. We construct n_k inductively. Start by choosing n_1 so that $|a_{n_1}| < 1$ (which we can since $a_n \to 0$). Then having chosen n_k, choose $n_{k+1} > n_k$ so that $|a_{n_{k+1}}| < (k+1)^{-2}$, which again we can do since $a_n \to 0$. Then we have a subsequence such that $|a_{n_k}| < k^{-2}$ and then by comparison with $\sum k^{-2}$ we see that $\sum a_{n_k}$ converges absolutely.

S32. Prove that the sequence $a_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log n \quad (n \in \mathbb{N})$ converges. [Hint: It is monotone and bounded.]

Solution. We have $a_{n+1} - a_n = \frac{1}{n} - \log(n+1) + \log n = \frac{1}{n} - \int_n^{n+1} \frac{1}{x} dx = \int_n^{n+1} (\frac{1}{n} - \frac{1}{x}) dx \geq 0$ since the integrand is ≥ 0 on the interval $[n, n+1]$. So (a_n) is increasing. On the other hand if we let $b_n = \frac{1}{n} + a_n$ the we find $b_{n+1} - b_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x} dx = \int_n^{n+1} (\frac{1}{n+1} - \frac{1}{x}) dx \leq 0$, so (b_n) is decreasing, and as $b_n > a_n$ it follows that both series are bounded and hence converge.

S33. Find the radius of convergence of the following power series.

(i) $\displaystyle\sum_n \frac{n^n}{n!} z^n$ (ii) $\displaystyle\sum_n n z^n$ (iii) $\displaystyle\sum_n \frac{z^n}{n!}$

(iv) $\displaystyle\sum_n (\log n) z^n$ (v) $\displaystyle\sum_n \frac{n}{4^n + n} z^n$.

Solution. (i) $\frac{a_n}{a_{n+1}} = (1 + \frac{1}{n})^{-n} \to e^{-1}$ so $R = e^{-1}$.
(ii) $\frac{a_n}{a_{n+1}} = \frac{n}{n+1} \to 1$ so $R = 1$.
(iii) $\frac{a_n}{a_{n+1}} = n + 1 \to \infty$ so $R = \infty$.
(iv) $\frac{a_n}{a_{n+1}} = \frac{\log n}{\log(n+1)} \to 1$ so $R = 1$.
(v) $\frac{a_n}{a_{n+1}} = \frac{n(4^{n+1}+n+1)}{(n+1)(4^n+n)} \to 4$ so $R = 4$.

E34. Can a power series of the form $\displaystyle\sum_n a_n (z-2)^n$ converge at $z = 0$ but diverge at $z = 2 + i$?

Solution. No. If it converges at 0 then $R \geq 2$ but if it diverges at $2 + i$ then $R \leq 1$.

S35. Let (a_n) and b_n be two sequences of complex numbers. For $n \in \mathbb{N}$ define $B_n = b_1 + \cdots + b_n$, and set $B_0 = 0$. Prove that for any positive integers N , M with $N > M$,

$$\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n (B_n - B_{n-1}) = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

Solution. Multiply out the $a_n(B_n - B_{n-1})$ terms and collect the terms with the same B_n.

H36. Consider the power series

$$\sum_n n z^n \quad , \quad \sum_n \frac{z^n}{n^2} \quad , \quad \sum_n \frac{z^n}{n}$$

Show that in all three cases the radius of convergence is $R = 1$. Show that the first converges at no point on the unit circle, the second converges at all points on the unit circle and the third converges at all points on the unit circle except $z = 1$.

Solution. In each case $\frac{a_n}{a_{n+1}}$ is a power of $\frac{n+1}{n}$ so $\to 1$, hence $R = 1$. If $|z| = 1$ then $|nz^n| = n$ so the first series cannot converge. And $|\frac{z^n}{n^2}| = \frac{1}{n^2}$ so the second converges by comparison with $\sum \frac{1}{n^2}$.

For the third series, if $z = 1$ then we have $\sum \frac{1}{n}$ which diverges. If $|z| = 1$ but $z \neq 1$, we apply problem 33 with $a_n = \frac{1}{n}$ and $b_n = z^n$. Then $B_n = z + z^2 + \cdots + z^n = \frac{z(1-z^n)}{1-z}$ which implies that (B_n) is bounded (this is the part that fails if $z = 1$). We then have $\sum_{n=1}^{N} \frac{z^n}{n} = N^{-1}B_N + \sum_{n=1}^{N-1}(\frac{1}{n} - \frac{1}{n+1})B_n$. Now $N^{-1}B_N \to 0$ as $N \to \infty$, and $\sum(\frac{1}{n} - \frac{1}{n+1})$ is a convergent series of positive terms, so $\sum(\frac{1}{n} - \frac{1}{n+1})B_n$ converges by comparison, using the boundedness of (B_n). Hence $\sum \frac{z^n}{n}$ converges.

S37. Suppose that the power series $\sum_n a_n z^n$ has radius of convergence $R \in (0, \infty)$ and that all a_n are nonnegative real numbers. Prove that if the series converges at $z = R$ then it converges absolutely at all points on the circle $|z| = R$.

Solution. If the series converges at $z = R$ then $\sum a_n R^n$ converges. Then for any z with $|z| = R$ we have $|a_n z^n| = a_n R^n$ so $\sum a_n z^n$ converges absolutely.

P38. Show that if all a_n are positive integers then the radius of convergence of the power series $\sum_n a_n z^n$ is at most one.

solution. If $|z| = 1$ then $|a_n z^n| \geq 1$ for each n so the powers series cannot converge (as the terms do not tend to 0). So $R \leq 1$.

P39. Let (a_n) be a sequence of complex numbers. Show that the power series

$$\sum a_n z^n \quad , \quad \sum n a_n z^n \quad , \quad \sum a_n \frac{z^{n+1}}{n+1}$$

have the same radius of convergence.

Solution. Let R_1, R_2, R_3 be the respective radii of convergence. We show that $R_2 \geq R_3$, the other inequalities being proved in the same way (or more easily). So suppose $|z| < R_3$; then we show that $\sum n a_n z^n$ converges, so that $|z| \leq R_2$. To do this, choose K with $|z| < K < R_3$. Then $\sum |a_n| \frac{K^{n+1}}{n+1}$ converges, hence so does $\sum |a_n| \frac{K^n}{n+1}$. Now $n|a_n||z|^n = \{n(n+1)(|z|/K)^n\}\{|a_n|\frac{K^n}{n+1}\}$ and $n(n+1)(|z|/K)^n \to 0$ since $|z|/K < 1$, so for n large enough $n|a_n||z|^n \leq |a_n|\frac{K^n}{n+1}$ and it then follows by comparison that $\sum n|a_n||z|^n$ converges.

S40. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence of continuous functions which converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$. Let (x_n) be a sequence of real numbers which converges to a real number x. Show that $f_n(x_n) \to f(x)$.

Solution. Let $\epsilon > 0$. Then as f is continuous at x we can fine $\delta > 0$ so that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon/2$. Then we can find n_0 such that if $N \geq n_0$ then $|x_n - x| < \delta$ and hence $|f(x_n) - f(x)| < \epsilon/2$. Next we can find n_1 so that if $n \geq n_1$ then $|f_n(y) - f(y)| < \epsilon/2$ for all $y \in \mathbb{R}$. Now if $n \geq \max(n_0, n_1)$ then $|f_n(x_n) - f(x) \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$ as required.

S41. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence of continuous functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any sequence (x_n) of real numbers which converges to any real limit x, the sequence $(f_n(x_n))$ converges to $f(x)$.
(i) Does it follow that $f_n \to f$ uniformly?
(ii) Does it follow that $f_n \to f$ pointwise?

Solution. Yes to (ii), as for any given x we can take $x_n = x$ for all n and conclude that $f_n(x) \to f(x)$. But no to (i), as shown by the example $f_n(x) = x/n$ for which $f_n(x_n) = x_n/n \to 0$ for any convergent sequence (x_n) but (f_n) does not converge uniformly on $\mathbb{R}$.

S42. Prove that the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x\sqrt{n}}{1+nx^2}$ converges pointwise to the zero function. Is the convergence uniform over $\mathbb{R}$?

Solution. $f_n(0) = 0$ and if $x \neq 0$ then $|f_n(x)| < n^{-1/2}|x|^{-1} \to 0$ so $f_n \to 0$ pointwise. But $f_n(n^{-1/2}) = \frac{1}{2}$ so $\sup_{\mathbb{R}} |f_n(x) - 0| \geq \frac{1}{2}$ hence the convergence is not uniform on $\mathbb{R}$.

S43. Consider the sequence (f_n) of functions defined on $[0, 1)$ by $f_n(x) = nx^n$. Show that $f_n \to 0$ pointwise but $\int_0^1 f_n(x)dx \to 1$.

Solution. For $0 \leq x < 1$, $nx^n \to 0$ so $f_n \to 0$ pointwise on $[0, 1)$ but $\int_0^2 f_n(x)dx = \frac{n}{n+1} \to 1$.

S44. Consider the sequence of functions on $\mathbb{R}$ given by $f_n(x) = (x - \frac{1}{n})^2$. Prove that it converges pointwise and find the limit function. Is the convergence uniform on $\mathbb{R}$? Is the convergence uniform on bounded intervals?

Solution. For each $x \in \mathbb{R}$ we have $f_n(x) \to x^2$ so f_n converges pointwise with limit $f(x) = x^2$. We have $f_n(x)(x) - f(x) - \frac{2x}{n} + n^2$ which is unbounded so convergence is not uniform on $\mathbb{R}$. On an interval $[-K, K]$ we have $\sup_{[-K,K]} |f_n - f| = \sup |\frac{2x}{n} + n^{-2}| \leq \frac{2K}{n} + n^{-2} \to 0$ as $n \to \infty$, so convergence is uniform on $[-K, K]$ for any $K > 0$, and hence on any bounded interval.

S45. Let $f_n(x) = x - x^n$. Prove that f_n converges pointwise on $[0, 1]$ and find the limit function. Is the convergence uniform on $[0, 1]$? Is the convergence uniform on $[0, 1)$?

Solution. If $0 \leq x < 1$ then $x^n \to 0$ so $f_n(x) \to x$. And $f_n(1) = 0$. So f_n converges pointwise on $[0, 1]$ to f defined by $f(x) = x$ for $0 \leq x < 1$ and $f(1) = 0$. On $[0, 1)$ we have $\sup_{[0,1)} |f_n(x) - f(x)| = \sup_{[0,1)} x^n = 1 \not\to 0$ so the convergence is not uniform on $[0, 1)$.

S46. Consider the sequence of functions defined on $[0, \infty)$ by $f_n(x) = \frac{x^n}{1+x^n}$.
(i) Prove that (f_n) converges pointwise and find the limit function.
(ii) Is the convergence uniform on $[0, \infty)$?
(iii) Is the convergence uniform on bounded intervals of the form $[0, a)$?

Solution. If $0 \leq x < 1$ then $x^n \to 0$ so $f_n(x) \to \frac{0}{1+0} = 0$. If $x = 1$ then $f_n(x) = \frac{1}{2}$. If $x > 1$ then $f_n(x) = \frac{1}{1+x^{-n}}$ and $x^{-n} \to 0$ so $f_n(x) \to 1$. So f_n converges pointwise on $[0, \infty)$ to f defined by $f(x) = 0$ for $0 \leq x < 1$, $\frac{1}{2}$ for $x = 1$, and 1 for $x > 1$.
(ii) No, because f is not continuous on $[0, \infty)$, being discontinuous at 1.
(iii) No if $a \geq 1$, because f is not continuous on $[0, a]$. If $0 \leq a < 1$ the convergence is uniform on $[0, a]$ because then $\sup_{[0,a]} \frac{x^n}{1+x^n} \leq a^n \to 0$.

S47. Let $f_n(x) = \frac{nx}{1+n^2x^2}$.
(i) Prove that (f_n) converges pointwise on $\mathbb{R}$ and find the limit function.
(ii) Is the convergence uniform on $[0, 1]$?
(iii) Is the convergence uniform on $[1, \infty)$?

Solution. (i) $f_n(0) = 0$ and if $x \neq 0$ then $|f_n(x)| \leq n|x| \to 0$ so $f_n(x) \to 0$. So f_n converges pointwise to 0 on $\mathbb{R}$.
(ii) No - on $[0, 1]$ we have $\sup_{[0,1]} |f_n(x)| \geq |f_n(\frac{1}{n}) = \frac{1}{2}$ so convergence is not uniform.
(iii) Yes - for $[1, \infty)$ we have $\sup_{[1,\infty)} |f_n(x)| \leq \frac{1}{n} \to 0$ so convergence is uniform.

S48. Define f_n on $[0, \infty)$ by $f_n(x) = \frac{nx}{1+nx}$
(i) Prove that (f_n) converges pointwise and find the limit function.
(ii) Is the convergence uniform on $[0, 1]$?
(iii) Is the convergence uniform on $(0, 1]$?
(iv) Is the convergence uniform on $[1, \infty)$?

Solution. (i) $f_n(0) = 0$ and for $x > 0$, $f_n(x) = \frac{1}{1+n^{-1}x^{-1}} \to 1$. So f_n converges pointwise on $[0, \infty)$ to f defined by $f(0) = 0$ and $f(x) = 1$ for $x > 0$.

(ii) No, as f is not continuous at 0.
(iii) No, as $\sup_{(0,1]} |f_n(x) - f(x)| = \sup_{(0,1]}(1 + nx)^{-1} = 1$.
(iv) Yes, as $\sup_{[1,\infty)}(1 + nx)^{-1} \leq \frac{1}{n} \to 0$.

E49. Find $\lim\limits_{n\to\infty} \displaystyle\int_0^1 \frac{2n + \sin x}{3n + \cos^2 x} dx$.

Solution. We have $f_n(x) = \frac{2+n^{-1}\sin x}{3+n^{-1}\cos x} \to \frac{2}{3}$ as $n \to \infty$, and the convergence is uniform since $|f_n(x) - \frac{2}{3}| = |n^{-1}\frac{3\sin x - 2\cos x}{(2+n^{-1}\sin x)(3+n^{-1}\cos x)}| \leq \frac{5}{2n} \to 0$. So $\int_0^1 f_n(x)dx \to \int_0^1 \frac{2}{3}dx = \frac{2}{3}$.

S50.(i) Prove that the sequence of functions $f_n(x) = nx(1 - x^2)^n$ converges pointwise on $[0, 1]$ and find the limit function. (You may use without proof that: $na^n \to 0$ for $a \in (0, 1)$).
(ii) Is the convergence uniform on $[0, 1]$? (Hint: Consider the integrals $\int_0^1 f_n$.)
(iii) Is the convergence uniform on $[a, 1]$ where $0 < a < 1$?

Solution. (i) $f_n(0) = 0$ and if $0 < x \leq 1$ then $|1-x^2| < 1$ so $n(1-x^2)^n \to 0$. So for every $x \in [0, 1]$ we have $f_n(x) \to 0$, i.e. f_n converges pointwise to 0 on $[0, 1]$.
(ii) $\int_0^1 f_n(x)dx = \frac{n}{2}\int_0^1 u^n du = \frac{n}{2(n+1)} \to \frac{1}{2}$ as $n \to \infty$, where we have used the substitution $u = 1-x^2$ to evaluate the integral. So f_n cannot converge uniformly to 0 on $[0, 1]$.
(iii) Yes, because $\sup_{[a,1]} |f_n(x)| \leq n(1 - a^2)^n \to 0$ as $n \to \infty$.

S51. If $f_n \to f$ and $g_n \to g$ uniformly on $\mathbb{R}$, does it follow that $f_n g_n \to fg$ uniformly on $\mathbb{R}$?

Solution. Not in general - for example if $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ then f_n converges uniformly to $f(x) = x$ and g_n to $g(x) = 0$ but $f_n(x)g_n(x) = \frac{x}{n}$ which does not converge uniformly to 0 (since $\sup_n \frac{x}{n} = \infty$).

If however f and g are both bounded, then uniform convergence of $f_n g_n$ does follow. For suppose $|f(x)| \leq M$ and $|g(x)| \leq M$ for all x. Let $\epsilon > 0$. Put $\delta = \min(1, \frac{\epsilon}{2M+1})$. Then we can find n_0 such that $|f_n(x) - f(x)| < \delta$ and $|g_n(x) - g(x)| < \delta$ for all $n \geq n_0$ and $x \in \mathbb{R}$. Then $|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)|(g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \leq (M+\delta)\delta + M\delta \leq (2M+1)\delta = \epsilon$ for all $n \geq n_0$ and $x \in \mathbb{R}$, showing that $f_n g_n \to fg$ uniformly on $\mathbb{R}$.

E52. Let $f_n(x) = \sqrt{x^2 + n^{-2}}$. Show that the sequence converges uniformly over $\mathbb{R}$ and find the limit function f.

Show that each f_n is differentiable but that f is not differentiable.

Solution. $f_n(x) \to \sqrt{x^2} = |x| = f(x)$ for all $x \in \mathbb{R}$. We have $|f_n(x) - f(x)| = \frac{n^{-2}}{|x|+\sqrt{x^2+n^{-2}}} \leq \frac{n^{-2}}{n^{-1}} = n^{-1} \to 0$ so $f_n \to f$ uniformly on $\mathbb{R}$.

Since $x^2 + n^{-2}$ is always positive, and $u \to u^{1/2}$ is differentiable for $u > 0$, f_n is differentiable on $\mathbb{R}$. f is not differentiable at 0.

E53. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = n^{-2}\sin(n^2 x)$. Show that the sequence converges uniformly on $\mathbb{R}$ and find the limit function f. Show also that each f_n is differentiable. Show that the sequence of the derivatives (f_n') does not converge (even pointwise) on $\mathbb{R}$.

Solution. $|f_n(x)| \leq n^{-2}$ so $f_n \to 0$ uniformly on $\mathbb{R}$. Clearly f_n is differentiable and $f_n'(x) = \cos(n^2 x)$. We have $f_n'(\pi) = \cos(n^2\pi) = (-1)^n$, which does not converge, so f_n' does not converge pointwise on $\mathbb{R}$.

S54. Find the subset of $(0, \infty)$ on which the sequence of functions $f_n(x) = n|\log x|^n$ converges pointwise. Is the convergence uniform?

Solution. If $a \geq 0$ then the sequence (na^n) converges iff $a < 1$. So the required set is the set of $x \in (0, \infty)$ such that $|\log x| < 1$, which is (e^{-1}, e), and on this set the pointwise limit is 0. We have $\sup_{(e^{-1},e)} f_n = n$ so the convergence is not uniform.

H55. Let $\alpha \in \mathbb{R}$. Find the largest subset of $\mathbb{R}$ on which the sequence of functions $f_n(x) = \frac{xn^\alpha}{e^{nx}\log n}$ $(n \geq 2)$ converges pointwise. Is the convergence uniform on that set?

Solution. If $x < 0$ then e^{-nx} grows faster than any power of n as $n \to \infty$, so $f_n(x) \to \infty$. For $x = 0$ we have $f_n(0) = 0$ for all n. For $x > 0$, e^{-nx} decays faster than any power of n so $f_n(x) \to 0$. hence the convergence set is $[0, \infty)$ and the pointwise limit on this set is $f(x) = 0$.

The function xe^{-nx} has derivative $(1 - nx)e^{-nx}$ so it is increasing for $x < \frac{1}{n}$ and decreasing for $x > \frac{1}{n}$, and hence its maximum is its value at $x = \frac{1}{n}$, namely $\frac{e^{-1}}{n}$. So $\sup_{[0,\infty)} |f_n(x)| = \frac{e^{-1}n^{\alpha-1}}{\log n}$. This $\to \infty$ if $\alpha > 1$ (see problem 10(a)) and $\to 0$ if $\alpha \leq 1$. So the convergence is uniform on $[0, \infty)$ iff $\alpha \leq 1$.

S56. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function. Suppose that there is an $L \in (0, 1)$ such that for all $x \in [-1, 1]$, $|f(x)| \leq L|x|$ Define a sequence of functions on $[-1, 1]$ by $f_1 = f$ and $f_n(x) = f(f_{n-1}(x))$. Show that this sequence converges uniformly to the zero function.

Solution. Let $M_n = \sup_{[-1,1]} |f_n(x)|$. Then for all $x \in [-1, 1]$, $|f_n(x)| = |f(f_{n-1}(x))| \leq L|f_{n-1}(x)| \leq LM_{n-1}$. And $M_1 = \sup |f(x)| \leq L$ so by induction $M_n \leq L^n$. So $M_n \to 0$, hence $f_n \to 0$ uniformly on $[-1, 1]$.

H57. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions which converges uniformly on the *open* interval (a, b) to a function $f : (a, b) \to \mathbb{R}$. Show that the sequences $(f_n(a))$ and $(f_n(b))$ converge. Call their limits A and B respectively. Then show that the sequence of functions (f_n) converges uniformly on the *closed* interval $[a, b]$ to the function F defined on $[a, b]$ by $F(x) = f(x)$ for $x \in (a, b)$ and $F(a) = A$, $F(b) = B$.

Solution. The sequence f_n is uniformly Cauchy on (a, b). So, given $\epsilon > 0$, we can find n_0 such that for $n, m \geq n_0$ we have $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in (a, b)$. But $f_n - f_m$ is a continuous function, so we deduce that $|f_n(x) - f_m(x)| \leq \epsilon$ on $[a, b]$, so (f_n) is uniformly Cauchy on $[a, b]$. Then by Cauchy's criterion f_n converges uniformly on $[a, b]$ to a limit F and we must have $F(x) = f(x)$ on (a, b), and also F is the limit of f_n at the end-points.

H58. (Dini's Theorem) Let (f_n) be a decreasing sequence of continuous functions on the interval $[a, b]$ which converges pointwise to zero. Show that (f_n) converges uniformly to zero.

Solution. Let $\epsilon > 0$. We need to find n such that $f_n(x) < \epsilon$ for all $x \in [a, b]$ (and then the same will be true of f_m for all $m > n$ by the decreasing property). Suppose no such n exists. Split $[a, b]$ into two equal intervals I, I'. If we can find n such that $f_n(x) < \epsilon$ on I and n' such that $f_{n'} < \epsilon$ on I', then taking $m = \max(n, n')$ we would have $f_m(x) < \epsilon$ on $[a, b]$, a contradiction. So we can find an interval I_1, being either I or I', such that there is no n satisfying $F_n(x) < \epsilon$ for all $x \in I_1$.

The we can repeat this process, obtaining a sequence of closed intervals $I_1, I_2, \cdots$, such that I_{k+1} is half of I_k, and such that for each k there is no n such that $f_n(x) < \epsilon$ for all $x \in I_k$. By the nested intervals theorem $\cap I_k$ contains a single point x. Since $f_n(x) \to 0$ there is n such that $f_n(x) < \epsilon$. But as f_n is continuous this means that for k large enough $f_n(x) < \epsilon$ on I_k, which is a contradiction. This proves that we can indeed find n such that $f_n(x) < \epsilon$ for all $x \in [a, b]$, and hence $f_n \to 0$ uniformly.

S59. Show that the series of functions $\sum_{n=1}^{\infty} ne^{-nx}$ converges uniformly on $[1, \infty)$ to a (continuous) function f. [Hint: Use the Weierstrass M-test]. Find $\int_1^2 f(x)dx$.

Solution. We have $ne^{-nx} \le ne^{-n}$ for $x \in [1, \infty)$ and $\sum ne^{-n}$ converges by the ratio test, because $\frac{(n+1)e^{-(n+1)}}{ne^{-n}} = (1 + \frac{1}{n})e^{-1} \to e^{-1} < 1$. So $\sum ne^{-nx}$ converges uniformly by the M-test.

Then $\int_1^2 f(x)dx = \sum_{n=1}^{\infty} \int_1^2 ne^{-nx}dx = \sum_{n=1}^{\infty}(e^{-n} - e^{-2n}) = \frac{e^{-1}}{1-e^{-1}} - \frac{e^{-2}}{1-e^{-2}} = \frac{1}{e-1} - \frac{1}{e^2-1} = \frac{e}{e^2-1}$.

S60. Consider the series of functions $\sum_{n=0}^{\infty} 2^{1-n}x^n$ for $x \in (-2, 2)$. Prove that it converges pointwise and find the limit function. Is the convergence uniform on $(-2, 2)$? Let $\delta \in (0, 2)$. Is the convergence uniform on $(-\delta, \delta)$?

Solution. This is a geometric series $2\sum_{n=0}^{\infty}(\frac{x}{2})^n$ which converges for $x \in (-2, 2)$ to $S(x) = \frac{2}{1-x/2} = \frac{4}{2-x}$. The sum to n terms is $S_n(x) = \frac{2(1-(x/2)^n}{1-(x/2)}$ so $|S(x) - S_n(x)| = \frac{4(x/2)^n}{2-x}$. Then $\sup_{(-2,2)} |S(x) - S_n(x)| = \infty$ so the convergence is not uniform on $(-2, 2)$. But if $\delta \in (0, 2)$ then $\sup_{(-\delta,\delta)} |S(x) - S_n(x)| = 4(\delta/2)^n(2 - \delta)^{-1} \to 0$ so the convergence is uniform on $(-\delta, \delta)$.

S61. Consider the series of functions $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, which converges pointwise to the exponential function. Show that the convergence is uniform on every bounded interval. Is the convergence uniform on $\mathbb{R}$?

Solution. We have $\sup_{[-M,M]} |\frac{x^n}{n!} = \frac{M^n}{n!}$ and $\sum \frac{M^n}{n!}$ converges so the series converges uniformly on $[-M, M]$ for any M. But the convergence is not uniform on $\mathbb{R}$ as the individual terms are not bounded on $\mathbb{R}$.

S62. Is the function f defined on $[0, 1]$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Riemann integrable?

Solution. No. One can show this by calculating the upper and lower integrals. Let $P = (0 = x_1 < x_1 < \cdots < x_n = 1)$ be a partition. Claim: $M_i = \sup_{[x_{i-1},x_i]} f(x) = x_i$. To show this suppose $x_{i-1} \le c < x_i$. then there is a rational $y \in (c, x_i)$ by the density of the rationals for $f(y) = y > c$ and so $M_i > c$. Since c can be arbitrarily close to x_i we deduce $M_i \ge x_i$, and the reverse inequality also holds since $f(y) \le y \le x_i$ for all $y \in [x_{i-1}, x_i]$, so the claim is proved.

This is exactly the same M_i as we would get for the function $g(x) = x$, so we have $U(f, P) = U(g, P)$. This is true for every partition P and hence $\overline{\int_0^1} f = \overline{\int_0^1} g$ which, as shown in lectures, is $\frac{1}{2}$.

On the other hand by the density of the irrationals each interval $[x_{i-1}, x_i]$ contains y such that $f(y) = 0$, so $m_i = 0$ and $L(f, P) = 0$ for every partition P so $\underline{\int_0^1} f = 0$. Hence f is not Riemann integrable.

E63. Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable for which $|f|$ is Riemann integrable.

Solution. One example is as follows: let $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. Then if is not Riemann integrable (essentially this example was treated in lectures) but $|f(x)| = 1$ for all x so $|f|$ is integrable.

S64. Let $g : [a, b] \to \mathbb{R}$ be continuous and non-negative. If $\int_a^b g(x)dx = 0$, show that $g = 0$ on $[a, b]$.

Solution. Let $x \in [a, b]$ and suppose $g(x) > 0$. Let $\epsilon = g(x)/2$. Then there is $\delta > 0$ such that there is an interval $[c,d]$ of length $d-c=\delta$, containing x, and contained in $[a,b]$, such that $g(y)>\epsilon$ for $y\in[c,d]$. Then $\int_a^b g\geq\int_c^d g\geq\epsilon\delta>0$, a contradiction, so $g(x)=0$.

S65. Let $f,g:[a,b]\to\mathbb{R}$ be continuous and suppose that $g\geq 0$. Show that there exists a $\xi\in[a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$

Solution. Let $h(\xi)=\int_a^b f(x)g(x)dx - f(\xi)\int_a^b g(x)dx=\int_a^b\{f(x)-f(\xi)\}g(x)dx$. Let $y\in[a,b]$ be such that $f(y)=\sup_{[a,b]}f$. Then $f(x)-f(y)\leq 0$ for all $x\in[a,b]$ so $h(y)\leq 0$. Similarly using inf in place of sup we find $z\in[a,b]$ with $h(z)\geq 0$. Since h is continuous, by the intermediate value theorem we can find $\xi\in[a,b]$ with $h(\xi)=0$ as required.

S66. Let $f:[0,a]\to[0,b]$ be continuous, one-to-one and onto. Explain why

$$\int_0^a f(x)dx+\int_0^b f^{-1}(y)dy=ab$$

[Hint: There is a very simple geometric reason for this result]

Solution. Consider the rectangle in the (x,y) plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$. The curve $y=f(x)$ divides this rectangle into two regions, with areas $\int_0^a f(x)dx$ and $\int_0^b f^{-1}(y)dy$, so the sum of these areas must be the area of the rectangle, ab.

S67. Let $f,g:[a,b]\to\mathbb{R}$ be Riemann integrable. Prove the Cauchy-Schwarz inequality:

$$\int_a^b f(x)g(x)dx\leq\left(\int_a^b f(x)^2dx\right)^{\frac{1}{2}}\left(\int_a^b g(x)^2dx\right)^{\frac{1}{2}}$$

[Hint: $\int_a^b(f(x)-\lambda g(x))^2\,dx\geq 0$ for any constant $\lambda\in\mathbb{R}$]

Solution. The hint gives $\int f^2-2\lambda\int(fg)+\lambda^2\int g^2\geq 0$ and putting $\lambda=\int(fg)/\int g^2$ gives the result (unless $\int g^2=0$ in which case we have $\int f^2-2\lambda\int(fg)\geq 0$ and letting $\lambda\to\infty$ gives $\int(fg)=0$).

S68. Let $f:\mathbb{R}\to\mathbb{R}$ be locally Riemann integrable. Show that the function

$$g(x)=\int_0^x f(t)dt$$

is continuous.

Solution. Fix x. Then f is integrable on $[x-1,x+1]$ and so bounded on that interval, say $|f(y)|\leq M$ for $y\in[x-1,x+1]$. Then for such y we have $|g(y)-g(x)|=|\int_x^y f|\leq M|y-x|$ which shows that g is continuous at x.

H69. Let $f:[a,b]\to\mathbb{R}$ be continuous and let $M=\max\limits_{x\in[a,b]}|f(x)|$. Show that

$$\lim_{n\to\infty}\left(\int_a^b|f(x)|^n dx\right)^{1/n}=M$$

Solution. Let $a_n=\left(\int_a^b|f(x)|^n dx\right)^{1/n}$. We have $\int_a^b|f|^n\leq M^n(b-a)$ so $a_n\leq M(b-a)^{1/n}\to M$. On the other hand, given $\epsilon>0$ we can find $\delta>0$ and an interval of length δ on which $|f(x)|>M-\frac{\epsilon}{2}$, and then we have $\int_a^b |f|^n \geq \delta(M - \frac{\epsilon}{2})^n$ and so $a_n \geq \delta^{1/n}(M - \frac{\epsilon}{2}) \to M - \frac{\epsilon}{2}$. It follows from these two bounds for a_n that if n is large enough then $M - \epsilon < a_n < M + \epsilon$, and we deduce that $a_n \to M$ as required.

S70. Show that the function $f(x) = |x|^{1/2}$ is uniformly continuous on $\mathbb{R}$. (Hint: when estimating $f(x) - f(y)$, you may find it helpful to consider the case when x and y are close to 0 separately).

Solution. Let $\epsilon > 0$. If $|x| \leq \epsilon^2$ and $|y| \leq \epsilon^2$ then $0 \leq f(x) \leq \epsilon$ and the same for $f(y)$ so $|f(x) - f(y)| \leq \epsilon$. On the other hand, if $|x - y| \leq \epsilon^2$, and $|x|$ and $|y|$ are not both $\leq \epsilon^2$, then

$$|f(x) - f(y)| = \frac{||x| - |y||}{|x|^{1/2} + |y|^{1/2}} \leq \frac{|x - y|}{\epsilon} \leq \epsilon$$

So in any event, if $|x - y| \leq \epsilon^2$ then $|f(x) - f(y)| \leq \epsilon$, proving uniform continuity.

An alternative, slightly shorter argument is as follows: if $|x - y| < \epsilon^2$ then $(f(x) - f(y))^2 = (|x|^{1/2} - |y|^{1/2})^2 \leq \left||x|^{1/2} - |y|^{1/2}\right| (|x|^{1/2} + |y|^{1/2}) = |(|x| - |y|)| < \epsilon^2$ so $|f(x) - f(y)| < \epsilon$.

S71. Suppose f is a bounded function on a closed interval $[a, b]$, and that f is continuous on (a, b). Show that f is Riemann integrable on $[a, b]$, without using Theorem 3 of section 3.3.

Deduce that the function f defined on $[0, 1]$ by

$$f(x) = \begin{cases} \sin \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Solution. Let M be a bound for $|f|$, and let $\delta > 0$. Then f is R.I. on $[a + \delta, b - \delta]$. Let P be a partition of $[a + \delta, b - \delta]$, and let $\tilde{P}$ be the partition of $[a, b]$ obtained by adding the points a, b to P. Then $U(f, \tilde{P}) \leq U(f, P) + 2\delta M$ and so $\overline{\int_a^b} f \leq U(f, P) + 2\delta M$. Taking the inf over partitions P of $[a = \delta, b - \delta]$ we get $\overline{\int_a^b} f \leq \int_{a+\delta}^{b-\delta} f + 2\delta M$. Similarly $\underline{\int_a^b} f \geq \int_{a+\delta}^{b-\delta} f - 2\delta M$. So $\overline{\int_a^b} f - \underline{\int_a^b} f \leq 4\delta M$. This holds for all $\delta > 0$ so $\overline{\int_a^b} f = \underline{\int_a^b} f$ as required.

The second part follows since $|f(x)| \leq 1$ for all $x \in [0, 1]$ and f is continuous on $(0, 1]$.

E72. Show that for any two integers n, m the following orthogonality relations hold:

(i) $\displaystyle\frac{1}{2\pi} \int_0^{2\pi} \cos nx \cos mx dx = 0$ if $n \neq m$ and $\frac{1}{2}$ if $n = m$

(ii) $\displaystyle\frac{1}{2\pi} \int_0^{2\pi} \sin nx \ \sin mx \ dx = 0$ if $n \neq m$ and $\frac{1}{2}$ if $n = m$

(iii) $\displaystyle\frac{1}{2\pi} \int_0^{2\pi} \sin nx \ \cos mx \ dx = 0.$

Solution. They are all elementary integrations, using trig identities like $\cos A \cos B = \frac{1}{2}\{\cos(A - B) + \cos(A + B)\}$.

S73. Fix $x \in \mathbb{R}$. Show that:
(i) The sequence (e^{inx}) converges if and only if $x \in 2\pi\mathbb{Z}$.
(ii) The sequence $(\sin(nx))$ converges if and only if $x \in \pi\mathbb{Z}$.
(iii) The sequence $(\cos(nx))$ converges if and only if $x \in 2\pi\mathbb{Z}$.

Solution. (i) if $x = 2m\pi$ where $m \in \mathbb{Z}$ then $e^{inx} = 1$ for all n so the sequence converges. Conversely, if $e^{inx} \to z$ then $e^{inx} - e^{i(n+1)x} \to 0$. But $|e^{inx} - e^{i(n+1)x}| = |1 - e^{ix}|$ so it follows that $|1 - e^{ix}| = 0$ so x is a multiple of 2π.

(ii) If $x = m\pi$ then each term is 0. Conversely if the sequence converges then $\sin(n+2)x - \sin nx \to 0$ i.e. $2\sin x\cos(n+1)x \to 0$ so either $\sin x = 0$ or $(\cos nx)$ converges. In the first case $x \in \pi\mathbb{Z}$ and in the second $e^{inx} = \cos nx + i\sin nx$ converges, so $x \in 2\pi\mathbb{Z}$ by (i). In either case $x \in \pi\mathbb{Z}$.
(iii) If the sequences converges, then in the same way as (ii) (using $\cos nx - \cos(n+2)x = 2\sin x\sin(n+2)x)$ we get $x \in \pi\mathbb{Z}$. If x is an odd multiple of π then $(\cos nx)$ alternates between ± 1 so we must have $x \in 2\pi\mathbb{Z}$.

S74. Let f be defined on $[-\pi, \pi)$ by $f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 0 & 0 \le x < \pi \end{cases}$
and extended to be 2π-periodic. Find the Fourier series of f in real form.

Solution. We have $a_k = \frac{1}{\pi}\int_{-\pi}^{0} x\cos kx dx = \frac{1}{\pi k}[x\sin kx]_{-\pi}^{0} - \frac{1}{\pi k}\int_{-\pi}^{0}\sin kx dx = 0 + \frac{1}{\pi k^2}[\cos kx]_{-\pi}^{0} = \frac{1-(-1)^k}{\pi k^2} = \frac{2}{\pi k^2}$ if k is odd and 0 if k is even and $k \neq 0$. Also $a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x dx = -\frac{\pi}{2}$.

And $b_k = \frac{1}{\pi}\int_{-\pi}^{0} x\sin kx dx = -\frac{1}{\pi k}[x\cos kx]_{-\pi}^{0} + \frac{1}{\pi k}\int_{-\pi}^{0}\cos kx dx = -\frac{(-1)^k}{k}$.

Hence the Fourier series is $-\frac{\pi}{4} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2} - \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin kx$.

S75. Let f be defined on $[-\pi, \pi)$ by $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$
and extended to be 2π-periodic. Let g be the convolution $g = f * f$. Show that $g(x) = |x|$ for $x \in [-\pi, \pi]$.

Then calculate the Fourier coefficients $c_n(f)$ and $c_n(g)$, and verify that $c_n(g) = 2\pi c_n(f)^2$.

Solution. We have $g(x) = \int_{-\pi}^{\pi} f(x-y)f(y)dy = \int_{0}^{\pi} f(x-y)dy$. Now if $x \in [0, \pi]$ and $y \in (0, \pi)$ then $f(x-y) = 1$ iff $y \le x$. Thus $g(x) = x$ for $x \in [0, \pi]$ and in a similar way we find $g(x) = -x$ for $x \in [-\pi, 0]$ so $g(x) = |x|$ for $x \in [-\pi, \pi]$.

Then we have $c_n(f) = \frac{1}{2\pi}\int_{0}^{1} e^{-inx}dx = \frac{1}{2\pi in}(1-(-1))^n = \frac{1}{i\pi n}$ if n is odd, and 0 if n is even, not 0. And $c_0 = \frac{1}{2\pi}\int_{0}^{\pi} dx = \frac{1}{2}$. For g we have $c_n(g) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|e^{-inx}dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx = \frac{1}{n\pi}[x\sin nx]_{0}^{\pi} - \frac{1}{n\pi}\int_{0}^{\pi}\sin nx dx = 0 - \frac{1}{n^2\pi}[\cos nx]_{0}^{\pi} = -\frac{2}{n^2\pi}$ if n is odd, and 0 if n is even, $n \neq 0$. Finally $c_0(g) = \frac{1}{\pi}\int_{0}^{\pi} x dx = \frac{\pi}{2}$. In each case we have $c_n(g) = 2\pi c_n(f)^2$.

S76. If $0 < r < 1$ and $x \in \mathbb{R}$, show that

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{inx} = \frac{1-r^2}{1-2r\cos x + r^2}$$

[Hint: Set $z = re^{ix}$ and sum the resulting geometric series.]

Solution. $\sum_{n=-\infty}^{\infty}|r|^n e^{inx} = \sum_{n=0}^{\infty}(re^{ix})^n + \sum_{n=0}^{\infty}(re^{-ix})^n - 1 = \frac{1}{1-re^{ix}} - \frac{1}{1-re^{-ix}} - 1 = \frac{1-re^{-ix}+1-re^{ix}}{(1-re^{ix})(1-re^{-ix})} - 1 = \frac{2-2r\cos x}{1-2r\cos x + r^2} - 1 = \frac{1-r^2}{1-2r\cos x + r^2}$.

S77. Show that $\sin x + \sin 2x + \cdots + \sin Nx = \frac{\cos\frac{x}{2} - \cos\left(N+\frac{1}{2}\right)}{2\sin\frac{x}{2}}$.

[Hint: Multiply both sides by $2\sin\frac{x}{2}$.]

Solution. We have $\sum_{n=1}^{N} 2\sin\frac{x}{2}\sin nx = \sum_{n=1}^{N}\{\cos(n-\frac{1}{2})x - \cos(n+\frac{1}{2})x\} = \cos\frac{x}{2} - \cos(N+\frac{1}{2})x$ and the result follows.

E78. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and periodic. Show that the following are equivalent:
(i) f is odd.
(ii) for all $n \in \mathbb{Z}$, $c_{-n} = -c_n$.
(iii) for all $n \in \{0, 1, 2, \ldots\}$, $a_n = 0$.

Solution. In general we have $c_{-n} + c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{inx} + 2^{-inx})f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = a_n$

so (ii) is equivalent to (iii). Now if f is odd then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0$ as the integrand is odd. Conversely if $a_n = 0$ and we define $g(x) = f(x) + f(-x)$ then g is even so $b_n(g) = 0$ and $a_n(g) = 2a_n(f) = 0$ so all the Fourier coefficients of g are 0, which implies $g = 0$ and hence f is odd. So (i) is also equivalent to (iii).

E79. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and periodic. Show that the following are equivalent:
(i) f is real valued.
(ii) for all $n \in \mathbb{Z}$, $\overline{c_n} = c_{-n}$.
(iii) for all $n \in \{0, 1, 2, \ldots\}$, $a_n, b_n \in \mathbb{R}$.

Solution. Writing $e^{-inx} = \cos nx - i \sin nx$ in the definition of c_n we get $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ for $n \geq 0$ so $\bar{c}_n - c_{-n} = \frac{1}{2}(\bar{a}_n - a_n) + \frac{i}{2}(\bar{b}_n - b_n) = -i\Im a_n + \Im b_n$ so that (ii) is equivalent to (iii). Now if f is real then a_n and b_n are real from their definitions. Conversely we have $a_n(\overline{f}) = \bar{a}_n$ and the same for b_n so if a_n and b_n are all real then f and $\overline{f}$ have the same Fourier coefficients, so $f = \overline{f}$ and f is real. Hence (i) is equivalent to (iii).

S80. Suppose that the trigonometric polynomial $f(x) = \sum_{n=-N}^{N} c_n e^{inx}$ vanishes at x_0. Show that there exists another trigonometric polynomial $g(x)$ such that $f(x) = (e^{ix} - e^{ix_0})\, g(x)$.

Solution. $f(x) = f(x) - f(x_0) = \sum_{n=-N}^{N} c_n e^{inx} - \sum_{n=-N}^{N} c_n e^{inx_0} = \sum_{n=-N}^{N} c_n(e^{inx} - e^{inx_0})$. Then it suffices to find for each $n \in \{-N, \cdots, N\}$ a trigonometric polynomial g_n such that $e^{inx} - e^{inx_0} = (e^{ix} - e^{ix_0})g_n(x)$. This is trivial if $n = 0$. Now using the identity $a^n - b^n = (a - b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$ we have $e^{inx} - e^{inx_0} = (e^{ix} - e^{ix_0})\sum_{k=0}^{n-1} e^{i(n-1-k)x_0} e^{ikx}$ for $n > 0$. And if $n < 0$, say $n = -m$, we similarly find $e^{-imx} - e^{-imx_0} = (e^{-ix} - e^{-ix_0})\sum_{k=0}^{m-1} e^{-i(m-1-k)x_0} e^{-ikx} = -(e^{ix} - e^{ix_0})\sum_{k=0}^{m-1} e^{-i(m-k)x_0} e^{-i(k+1)x}$ giving the required trigonometric polynimials.

S81. Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and continuously differentiable. Show that for all integers n,

$$c_n(f') = inc_n(f)$$

Solution. $c_n(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \frac{1}{2\pi}[f(x)e^{-inx}]_{-\pi}^{\pi} + \frac{in}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0 + inc_n(f) = inc_n(f)$.

S82. Calculate the complex Fourier coefficients of the following periodic functions:
(i) $f(x) = \frac{\pi - x}{2}$, $0 < x < 2\pi$, and extended to be 2π-periodic.
(ii) $g(x) = \frac{(x-\pi)^2}{4}$, $0 \leq x \leq 2\pi$, and extended to be 2π-periodic.

Solution. (i) $c_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi - x}{2} e^{-inx}dx = \frac{1}{2\pi}[\frac{\pi - x}{-2in} e^{-inx}]_0^{2\pi} - \frac{1}{2\pi in}\int_0^{2\pi} e^{-inx}dx = -\frac{1}{2\pi in}(-\pi - \pi) - 0 = \frac{1}{in}$ for $n \neq 0$. And $c_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi - x}{2}dx = 0$.
(ii) $c_n(g) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} e^{-inx}dx = \frac{1}{2\pi}[\frac{(\pi - x)^2}{-4in} e^{-inx}]_0^{2\pi} - \frac{1}{2\pi in}\int_0^{2\pi} \frac{\pi - x}{2} e^{-inx}dx = 0 - \frac{1}{in}c_n(f) = \frac{1}{n^2}$ for $n \neq 0$, where f is as in part (i). And $c_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}dx = \frac{\pi^2}{12}$.

S83. Calculate the Fourier series of the following periodic functions in real form:
(i) $f(x) = |x|$, $-\pi \leq x < \pi$, and extended to be 2π-periodic.
(ii) $g(x) = |\sin x|$, $x \in \mathbb{R}$.

Solution. (i) As f is even, $b_k = 0$. And $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos kx dx = \frac{2}{\pi}\int_0^{\pi} x \cos kx dx = \frac{2}{\pi}[\frac{x \sin kx}{k}]_0^{\pi} - \frac{2}{\pi k}\int_0^{\pi} \sin kx dx = 0 + \frac{2}{\pi k^2}[\cos kx]_0^{\pi} = \frac{2}{\pi k^2}((-1)^k - 1) = -\frac{4}{\pi k^2}$ if k is odd and 0 if k is even, $k \neq 0$. Also $a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \pi$. So the Fourier series is $\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}(2m + 1)^{-2}\cos(2m + 1)x$.
(ii) again f is even so $b_k = 0$ and $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\cos kx dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx dx$
$= \frac{1}{\pi}\int_0^{\pi}\{\sin(k + 1)x - \sin(k - 1)x\}dx = \frac{1}{\pi}[-\frac{\cos(k+1)x}{k+1} + \frac{\cos(k-1)x}{k-1}]_0^{\pi} = \frac{1}{\pi}\{\frac{1-(-1)^{k+1}}{k+1} - \frac{1-(-1)^{k-1}}{k-1}\}$

$= \frac{2}{\pi}(\frac{1}{k+1} - \frac{1}{k-1}) = -\frac{4}{\pi(k^2-1)}$ if k is even, and 0 if k is odd and $k \neq 1$. And $a_1 = \frac{1}{\pi}\int_0^\pi \sin 2x dx = -\frac{1}{2\pi}[\cos 2x]_0^\pi = 0$. Then $a_0 = \frac{4}{\pi}$ and the Fourier series is $\frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^\infty \frac{\cos 2mx}{4m^2-1}$.

S84. Let $f : \mathbb{R} \to \mathbb{C}$ be periodic. Fix an integer m. Show that:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(mx)dx = \begin{cases} \hat{f}\left(\frac{n}{m}\right), & \text{if } n \text{ divides } m \\ 0, & \text{if } n \text{ doesn't divide } m \end{cases}$$

Solution. Putting $mx = u$ we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(mx)dx = \frac{1}{2\pi m}\int_{-m\pi}^{m\pi} e^{-nu/m} f(u)du$ $= \frac{1}{2m\pi}\int_{-m\pi}^{(2-m)\pi}\sum_{k=0}^{m-1} e^{-in(u+2k\pi)/m} f(u + 2k\pi)du = \frac{1}{2\pi}\int_{-m\pi}^{(2-m)\pi}\{\frac{1}{m}\sum_{k=0}^{m-1} e^{-2\pi ikn/m}\}e^{-inu/m} f(u)du$. Now if n/m is not an integer then $\sum_{k=0}^{m-1} e^{-2\pi ikn/m} = \frac{1-e^{-2\pi in}}{1-e^{2\pi in/m}} = 0$ while if n/m is an integer then $\frac{1}{m}\sum_{k=0}^{m-1} e^{-2\pi ikn/m} = 1$ and the result follows.

S85. Let $f : \mathbb{R} \to \mathbb{C}$ be periodic. Fix an integer m and define $g(x) = f(mx)$. If $f(x)$ has Fourier series $\sum_{n=-\infty}^\infty c_n e^{inx}$ show that $f(mx)$ has Fourier series $\sum_{n=-\infty}^\infty c_n e^{inmx}$.

Solution. This follows from problem 84.

S86. Find the Fourier series of the periodic function determined by $f(x) = x$, $-\pi < x \leq \pi$ and then use Parseval's identity to show that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution. $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}[\frac{xe^{-inx}}{-in}]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} e^{-inx}dx = -\frac{1}{2\pi in}\{\pi e^{-in\pi} - (-\pi)e^{in\pi}\} + 0 = -\frac{(-1)^n}{in}$ for $n \neq 0$ and $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$. So the Fourier series is $i\sum_{n\neq 0}\frac{(-1)^n}{n}e^{inx}$ (the real form is $-2\sum_{n=1}^\infty \frac{(-1)^n}{n}\sin nx$). Also $|c_n|^2 = |c_{-n}|^2 = \frac{1}{n^2}$ for $n \neq 0$ and $\frac{1}{2\pi}\int_\pi^\pi f^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \pi^2/3$ so Parseval gives $\pi^2/3 = \sum |c_n|^2 = 2\sum_{n=1}^\infty \frac{1}{n^2}$ and the result follows.

S87. Show that there is a unique sequence of polynomials $P_0, P_1, \cdots$ such that $P_0(x) = 1$, and $P_n'(x) = P_{n-1}(x)$ and $\int_{-\pi}^{\pi} P_n(x)dx = 0$ for $n \geq 1$.

Verify that $P_1(x) = x$ and $P_2(x) = \frac{x^2}{2} - \frac{\pi^2}{6}$, and find $P_3(x)$.

Now let $c_{n,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}P_n(x)dx$, the complex Fourier coefficients of P_n. Show that $c_{n,k} = \frac{1}{ik}c_{n-1,k}$ for $n > 1$ and $k \neq 0$. (Hint: first show that $P_n(\pi) = P_n(-\pi)$ for $n > 1$).

Calculate $c_{1,k}$ and deduce that $c_{n,k} = -\frac{(-1)^k}{(ik)^n}$ for $n \geq 1$ and $k \neq 0$.

Then use Parseval's relation to show that

$$\sum_{k=1}^\infty k^{-2n} = \frac{1}{4\pi}\int_{-\pi}^{\pi} P_n^2$$

for $n = 1, 2, \cdots$. Hence show that $\sum_{k=1}^\infty \frac{1}{k^6} = \frac{\pi^6}{945}$.

Solution. The existence of P_n is shown by induction - supposing P_{n-1} has been found, then let g be an antiderivative of P_{n-1}, which will be a polynomial. Then choose a constant C so that $\int_{-\pi}^{\pi}(g(x) + C)dx = 0$ - it is clear that there is a unique such C, namely $-\frac{1}{2\pi}\int_{-\pi}^{\pi} g$. Then we define $P_n(x) = g(x) + C$. Since any polynomial f such that $f' = P_{n-1}$ must be of the form $g + C$ for some constant C we see that there is no other choice for P_n.

For P_1 and P_2 as defined in the question one checkes that $P_1' = P_0$, $P_2' = P_1$ and both integrate to 0. For P_3, we start with $g(x) = \frac{x^3}{6} - \frac{\pi^2 x}{6}$ as an antiderivative of P_2 and then we find $\int_{-\pi}^{\pi} g = 0$ so in fact $P_3 = g$.

We have $P_n(\pi) - P_n(-\pi) = \int_{-\pi}^{\pi} P_n'(x)dx = \int_{-\pi}^{\pi} P_{n-1}(x)dx = 0$ if $n > 1$. Using this, we integrate by parts to get $c_{n,k} = \frac{1}{2\pi}[\frac{e^{-ikx}}{-ik}P_n(x)]_{-\pi}^{\pi} + \frac{1}{2\pi ik}\int_{-\pi}^{\pi} e^{-ikx}P_n'(x)dx = 0 + \frac{1}{2\pi ik}\int_{-\pi}^{\pi} e^{-ikx}P_{n-1}(x)dx = \frac{1}{ik}c_{n-1,k}$ for $n > 1$.

$c_{1,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-ikx}dx = -\frac{(-1)^k}{ik}$ for $k \neq 0$ (see problem 84). Then it follows from the last part that $c_{n,k} = -\frac{(-1)^k}{(ik)^n}$ for $n \geq 1$. We also have $c_{n,0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(x)dx = 0$ for $n \geq 1$.

Then for $n \geq 1$ we have $|c_{n,k}|^2 = |c_{n,-k}|^2 = k^{-2n}$ for $k \neq 0$, so $\sum_{k=\infty}^{\infty} |c_{n,k}|^2 = 2\sum_{k=1}^{\infty} k^{-2n}$ so Parseval gives $\sum_{k=1}^{\infty} k^{-2n} = \frac{1}{4\pi}\int_{-\pi}^{\pi} P_n^2$.

Taking $n = 3$ we get $\sum_{k=1}^{\infty} k^{-6} = \frac{1}{4\pi}\int_{-\pi}^{\pi}(\frac{x^3-\pi^2 x}{6})^2 dx$ which works out to $\frac{\pi^6}{945}$.

S88. Let $a, b \in \mathbb{R}$ with $a \neq 0$, and define f on $[-\pi, \pi)$ by $f(t) = e^{(a+ib)t}$. Calculate the complex Fourier coefficients c_k of f. By applying Parseval's formula, show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{a^2 + (b-k)^2} = \frac{\pi \sinh 2a\pi}{a(\cosh 2a\pi - \cos 2b\pi)}$$

Solution. We have $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a+ib-ik)t}dt = \frac{1}{2\pi(a+ib-ik)}\{e^{(a+ib-ik)\pi} - e^{-(a+ib-ik)\pi}\} = \frac{(-1)^k}{2\pi(a+ib-ik)}\{e^{(a+ib)\pi} - e^{-(a+ib)\pi}\} = \frac{(-1)^k}{\pi(a+ib-ik)}(\sinh a\pi \cos b\pi + i\cosh a\pi \sin b\pi)$ and so

$$|c_k|^2 = \frac{\sinh^2 a\pi \cos^2 b\pi + \cosh^2 a\pi \sin^2 b\pi}{\pi^2(a^2 + (b-k)^2)} = \frac{\cosh 2a\pi - \cos 2b\pi}{2\pi^2(a^2 + (b-k)^2)}$$

Now $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2at}dt = \frac{1}{2a\pi}\sinh 2a\pi$. Hence Parseval gives $\frac{\sinh 2a\pi}{2a\pi} = \frac{\cosh 2a\pi - \cos 2b\pi}{2\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{a^2+(b-k)^2}$ and the result follows.

S89. Let $a, b \in \mathbb{R}$ with $a \neq 0$, and define f on $[-\pi, \pi)$ by $f(t) = (x-b)e^{ax}$. Calculate the complex Fourier coefficients c_k of f.

Now suppose $a \neq 0$ is given. By choosing an appropriate b and applying Parseval's formula, show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(a^2 + k^2)^2} = \frac{\pi\{a\pi(\text{cosech } a\pi)^2 + \coth a\pi\}}{2a^3}$$

Solution. $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x-b)e^{(a-ik)x}dx = \frac{1}{2\pi}[\frac{(x-b)e^{(a-ik)x}}{a-ik}]_{-\pi}^{\pi} - \frac{1}{2\pi(a-ik)}\int_{-\pi}^{\pi} e^{(a-ik)x}dx = \frac{(-1)^k}{2\pi(a-ik)}\{(\pi - b)e^{a\pi} - (-\pi - b)e^{-a\pi}\} - \frac{1}{2\pi(a-ik)^2}[e^{(a-ik)x}]_{-\pi}^{\pi} = \frac{(-1)^k}{\pi(a-ik)}\{\pi\cosh a\pi - b\sinh a\pi\} - \frac{(-1)^k \sinh a\pi}{\pi(a-ik)^2}$.

The second term has the right form to give the required series so we choose $b = \pi\coth a\pi$ to eliminate the first term, and then $c_k = -\frac{(-1)^k \sinh a\pi}{\pi(a-ik)^2}$ for all k, so $\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{\sinh^2 a\pi}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{(a^2+k^2)^2}$. And $\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x-b)^2 e^{2ax}dx = \frac{1}{2\pi}[(x-b)^2\frac{e^{2ax}}{2a}]_{-\pi}^{\pi} - \frac{1}{2\pi a}\int_{-\pi}^{\pi}(x-b)e^{2ax}dx = \frac{1}{4\pi a}\{((\pi-b)^2 e^{2a\pi} - (\pi+b)^2 e^{-2a\pi}\} - \frac{1}{4\pi a^2}\{(\pi - b)e^{2a\pi} + (\pi + b)e^{-2a\pi}\} + \frac{1}{8\pi a^3}(e^{2a\pi} - e^{-2a\pi}) = \frac{1}{2\pi a}\{(\pi^2 + b^2)\sinh 2\pi a - 2\pi b\cosh 2\pi a\} - \frac{1}{2\pi a^2}(\pi\cosh 2\pi a - b\sinh 2\pi a) + \frac{1}{4\pi a^3}\sinh 2\pi a$. Substituting $b = \pi\coth a\pi$ this simplifies to $\frac{1}{2a^2} + \frac{1}{4\pi a^3}\sinh 2\pi a$. So Parseval gives $\sum_{k=-\infty}^{\infty}\frac{1}{(a^2+k^2)^2} = \frac{\pi^2}{\sinh^2 a\pi}(\frac{1}{2a^2} + \frac{1}{4\pi a^3}\sinh 2\pi a)$ which gives the result.