Pure and Applied Analysis exam 2010, Semester 1 questions

1. **(a)** State the Bolzano-Weierstrass theorem.

Find all subsequential limits of the sequence $a_n = \frac{1}{n} + \sin(\frac{n\pi}{3})$. **[10 marks]**

(b) State, with justification, which of the following series converge:
(i) $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$, (ii) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$ and (iii) $\sum_{n=1}^{\infty} \frac{1}{n+3}$. **[8 marks]**

(c) Define the notion of *absolute convergence*. For each of the series in part (b) say whether the series converges absolutely. **[7 marks]**

2. **(a)** Define the notions of *pointwise convergence* and *uniform convergence* for sequences of functions. **[7 marks]**

(b) Let $f_n(x) = (x/n)^n e^{n-x}$. Show that (f_n) converges pointwise on $[0, \infty)$ and identify the limit function. By considering the derivative f_n', show that $\sup_{[0,\infty)} f_n(x) = 1$ and deduce that the convergence is not uniform on $[0, \infty)$.

Show that the convergence is uniform on $[0, a]$ for any $a > 0$. **[10 marks]**

(c) For each of the following two functions, state with reasons whether or not it is Riemann integrable on $[-1, 1]$:
(i) $f(x) = |x|$; (ii) $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$ **[8 marks]**

3. **(a)** Suppose that f is continuous on $[-\pi, \pi]$ and that $f(-\pi) = f(\pi)$, and also that f has a continuous bounded derivative on $(-\pi, \pi)$. Let $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)dx$ and $d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f'(x)dx$, the complex Fourier coefficients of f and f'.

Show that $d_n = inc_n$ for each n. **[6 marks]**

(b) State Parseval's relation, and use it to show that, if f is as in part (a), then the series $\sum_{n=-\infty}^{\infty} n^2 |c_n|^2$ converges. **[8 marks]**

(c) Now let $f(x) = x$ on $(-\pi, \pi)$. Calculate the Fourier coefficients c_n of f, and check that the series $\sum_{n=-\infty}^{\infty} n^2 |c_n|^2$ diverges in this case. Why does this not contradict the result of part (b)?
[11 marks]

Solutions

1. (a) Every bounded sequence of real numbers has a convergent subsequence.

As $\frac{1}{n} \to 0$, ths subsequential limits of (a_n) are the same as those of $b_n = \sin(\frac{n\pi}{3})$. We have $b_{6k} = b_{6k+3} = 0$, $b_{6k+1} = b_{6k+2} = \frac{\sqrt{3}}{2}$ and $b_{6k+4} = b_{6k+5} = -\frac{\sqrt{3}}{2}$ so 0 and $\pm\frac{\sqrt{3}}{2}$ are subsequential limits. If b_{n_k} is any convergent subsequence, then the sequence (n_k) must contain infinitely many numbers of the form $6j+r$ for some $r \in \{0,1,2,3,4,5\}$ so the limit must be one of $0, \pm\frac{\sqrt{3}}{2}$. So these 3 numbers are the only possible limits.

(b) (i) As $|\frac{n+1}{n^3}| \le \frac{2}{n^2}$ the series converges by comparison with $\sum n^{-2}$; (ii) writing $a_n = \frac{1}{\log n}$, the sequence (a_n) decreases and converges to 0, so by the alternating series test $\sum(-1)^n a_n$ converges; (iii) $\frac{1}{n+3} \ge \frac{1}{4n}$ so this series diverges by comparison with $\sum \frac{1}{n}$.

(c) The series $\sum a_n$ converges absolutely iff $\sum |a_n|$ converges. For the series in (b)(i), $|a_n| = a_n$ so the series converges absolutely. For (ii), $|a_n| = \frac{1}{\log n}$ so $\sum |a_n|$ diverges (e.g. by comparison with $\sum \frac{1}{n}$). (iii) does not converge, so does not converge absolutely.

2. (a) $f_n \to f$ pointwise on a set E iff, given $x \in E$ and $\epsilon > 0$, we can find n_0 such that for all $n > n_0$ we have $|f_n(x) - f(x)| < \epsilon$.

$f_n \to f$ uniformly on a set E iff, given $\epsilon > 0$, we can find n_0 such that for all $n > n_0$ and all $x \in E$ we have $|f_n(x) - f(x)| < \epsilon$.

(b) If $x \ge 0$ then $f_n(x) = e^{-x}(ex/n)^n \le e^{-x}2^{-n}$ for $n \ge 2ex$ so $f_n(x) \to 0$, so f_n converges pointwise with limit $f(x) = 0$. We have $f'(x) = n^{-n}e^{n-x}(nx^{n-1} - x^n)$ so f is increasing on $[0,n]$ and decreasing on $[n,\infty)$. Hence $\sup f_n(x) = f_n(n) = 1$. As this does not tend to 0 the convergence is not uniform on $[0,\infty)$.

For $x \in [0,a]$ we have $f_n(x) \le (ea/n)^n$ and $(ea/n)^n \to 0$ so $f_n \to 0$ uniformly on $[0,a]$.

(c) (i) f is a continuous function on the closed interval $[-1,1]$ and hence is R.I. on it; (ii) we show f is not R.I. by showing it is not R.I. on the subinterval $[\frac{1}{2},1]$. If I is any subinterval of $[\frac{1}{2},1]$ then I contains points where $f = 0$ and points where $f(x) = x \ge \frac{1}{2}$, so $\sup_I f \ge \frac{1}{2}$ and $\inf_I f = 0$. So for any partition of $[\frac{1}{2},1]$ the lower sum is 0 and the upper sum is $\ge \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. So $\underline{\int}_{1/2}^{1} f = 0$ and $\overline{\int}_{1/2}^{1} f \ge \frac{1}{4}$ so f is not R.I. on $[\frac{1}{2},1]$.

3. (a) $d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}f'(x)dx = \frac{1}{2\pi}[e^{-inx}f(x)]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}(-ine^{-inx})f(x)dx = 0 + inc_n$, using the fact that $f(\pi) = f(-\pi)$ for the first term, and so $d_n = inc_n$.

(b) Parseval: if f is continuous and 2π-periodic with Fourier coefficients c_n then $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$.

Applying Parseval to f' from (a) we get $\sum_{n=-\infty}^{\infty} |d_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx < \infty$ since f' is bounded. As $|d_n|^2 = n^2|c_n|^2$ it follows that $\sum_{n=-\infty}^{\infty} n^2|c_n|^2 < \infty$.

(c) $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}[\frac{xe^{-inx}}{-in}]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} e^{-inx}dx = -\frac{1}{2\pi in}\{\pi(-1)^n - (-\pi)(-1)^n\} + 0 = -\frac{(-1)^n}{in}$ provided $n \ne 0$. And $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx = 0$.

Then $|c_n|^2 = \frac{1}{n^2}$ for $n \ne 0$ and $|c_0|^2 = 0$. So $\sum_{n=-\infty}^{\infty} n^2|c_n|^2 = 2\sum_{n=1}^{\infty} n^2\frac{1}{n^2} = 2\sum_{n=1}^{\infty} 1$ which diverges. This does not contradict (b) as f does not satisfy $f(\pi) = f(-\pi)$.