1. (i) $\mathcal{A}$ is an algebra. First $\mathbb{N} \in \mathcal{A}$ is clear. Next if $A \in \mathcal{A}$, and $n \in A^c$, then $n \notin A$ and then $n+5 \notin A$, since otherwise $n = (n+5)-5$ would be in A, which it isn't, and if $n > 5$ then $n-5 \notin A$ since otherwise $n = (n-5)+5$ would be in A. So $n \in A^c$ implies $n+5 \in A^c$, and if $n > 5$ then $n - 5 \in A^c$. hence $A^c \in \mathcal{A}$.

Also if $A, B \in \mathcal{A}$ and $n \in A \cup B$, then $n \in$ either A or B (or both). Say $n \in A$ then $n+5 \in A$ so $n+5 \in A \cup B$, with the same conclusion if $n \in B$. In the same way $n-5 \in A \cup B$ if $n > 5$. So $A \cup B \in \mathcal{A}$.

(ii) Not an algebra: $[0, 1]$ is in $\mathcal{A}$ but its complement is not.

(iii) This is an algebra. Suppose $A \in \mathcal{A}$. Then if A is finite $\mathbb{R}\backslash A^c = A$ is finite, whereas if $\mathbb{R}\backslash A$ is finite then $A^c = \mathbb{R}\backslash A$ is finite, so in either case $A^c \in \mathcal{A}$.

Also if $A, B \in \mathcal{A}$, then either A and B are both finite or one of them has finite complement. In the former case, $A \cup B$ is finite. In the latter, if say A^c is finite, then $\mathbb{R}\backslash(A \cup B) \subseteq A^c$ is also finite. So in each case $\alpha \cup B \in \mathcal{A}$.

2. Let X be the maximum of the two numbers. The possible values of X are $1, 2, \cdots, 6$. The sets of the partition are then $A_1, \cdots, A_6$ where A_k is the set of pairs (i, j) with $\max(i, j) = k$. There are 6 sets.

3. For $i = 0, 1, 2, 3$ let A_i be the set of outcomes with exactly i heads. Then $\Omega = \cup_{i=0}^3 A_i$ is the required partition. We have $A_0 = \{TTT\}$, $A_1 = \{HTT, THT, TTH\}$, $A_2 = \{HHT, HTH, THH\}$, $A_3 = \{HHH\}$.

4. There is one algebra for each way of partitioning Ω into disjoint non-empty sets, and a simple count shows there are exactly 15 such partitions (1 with a single set of size 4, 4 with two sets of size 3 and 1, 3 with two sets of size 2, 6 with three sets of size 2-1-1, and one with 1-1-1-1).

5. It is clear that $\mathcal{A}$ is closed under finite unions, so it remains to show that $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$. To simplify notation let $\mathcal{D}$ denote the collection of all sets D such that either D or D^c belong to $\mathcal{B}$. Then if $A \in \mathcal{A}$ we can write $A = \cup_{i=1}^n C_n$ where $C_i \in \mathcal{C}$. Then $A^c = \cap_{i=1}^n C_i^c$ and we have $C_i^c = \cup_{j=1}^{k_i} D_{ij}$ where $D_{ij} \in \mathcal{D}$. Then $x \in A^c$ if and only if, for each $i = 1, 2, \cdots, n$, x belongs to at least one D_{ij} where $1 \le j \le k_i$. Then we can write $A^c = \cup(D_{1j_1} \cap \cdots \cap D_{nj_n})$ where the union is over all choices of $j_1, \cdots, j_n$ with $1 \le j_i \le k_i$.

6. (i) This is countable: let $r_1, r_2, \cdots$ be the rationals arranged in a sequence. Then for each n, the set A_n of triples (a, b, r_n) with $a, b \in \mathbb{Q}$ is countable as the set of pairs of rationals is countable (lectures), so $\cup_n A_n$ is countable

(ii) not countable - if it was countable then since $\mathbb{Q}$ is countable, $(0, 0.1)$ would be countable, which can be disproved by using decimal expansions (just as the proof in lectures that $\mathbb{R}$ is uncountable).

(iii) this is countable: let A_k be the set of all sequences (m_n) such that $m_n = 0$ for $n > k$. Then A_k is in one-one correspondence with the set of all k-tuples of integers, which is countable by the argument used to show (i). Then the given set is $\cup_{k=1}^\infty A_k$ which is countable.

7. Suppose $A \in \mathcal{C}$. Then if A is countable $\mathbb{R}\backslash A^c = A$ is countable, whereas if $\mathbb{R}\backslash A$ is countable then $A^c = \mathbb{R}\backslash A$ is countable, so in either case $A^c \in \mathcal{C}$.

Now suppose $A_1, A_2, \cdots \in \mathcal{C}$ and let $A = \cup_k A_k$. If all A_k are countable then A is the union of a sequence of countable sets and hence countable. Otherwise there exists k such that A_k^c is countable and then $A^c \subseteq A_k^c$ is also countable. In either case $A \in \mathcal{C}$. Hence $\mathcal{C}$ is a σ-algebra.

Then, if $\mathcal{A}$ is the algebra defined in 1(iii), $\mathcal{C}$ is a σ-algebra containing $\mathcal{A}$. On the other hand if $\mathcal{D}$ is any σ-algebra containing $\mathcal{A}$ then, as any countable set is a countable union of finite sets, $\mathcal{D}$ contains all countable sets, and then also their complements, so $\mathcal{C} \subseteq \mathcal{D}$. So $\mathcal{C}$ is the σ-algebra generated by $\mathcal{A}$.

8. Let $\mathcal{A}$ be the σ-algebra generated by the open intervals. It was shown in lectures that any open interval is in $\mathcal{B}$, hence $\mathcal{A} \subseteq \mathcal{B}$. On the other hand any interval of the form $[a, b)$ can be written as a countable intersection of open intervals $\cap_{n=1}^{\infty}(a - \frac{1}{n}, b)$ and so is in $\mathcal{A}$ and similarly for intervals like $[a, \infty)$ and $(-\infty, b)$. Since $\mathcal{B}$ is generated by such intervals we conclude that $\mathcal{A} = \mathcal{B}$.

9. We have $\Omega = A_{\mathbb{N}} \in \mathcal{A}$. If $A \in \mathcal{A}$ then $A = A_E$ for some E and then $A^c = A_{E^c} \in \mathcal{A}$. Finally if $B_1, B_2, \cdots \in \mathcal{A}$ then $B_i = A_{E_i}$ for some $E_i \subseteq \mathbb{N}$ and then $\cup_i B_= A_E \in \mathcal{A}$ where $E = \cup_i E_i$. So $\mathcal{A}$ is a σ-algebra.

Suppose $(A_1, A_2, \cdots)$ forms a partition of $\mathbb{R}$. Since $\mathbb{R}$ is uncountable at least one A_k must contain more than one point, say $x, y \in A_k$ with $x \neq y$. Now let B be an open interval containing x but not y. Then B is a Borel set but cannot be expressed in the form $\cup_{j \in E} A_j$ (for if it could, since $x \in B$ we must have $k \in E$ but then $y \in B$, a contradiction).

10. A is the set of x such that there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $|f_n(x)| \leq m$. So we have $A = \cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{mn}$ where $A_{mn} = \{x : |f_n(x)| \leq m\}$ is a closed set. Hence A is a Borel set.

11. Let $U = \cup_\alpha U_\alpha$ be the union of a family (U_α) of open sets. Let $x \in U$. Then $x \in U_\alpha$ for some α. Since U_α is open there is $\delta > 0$ such that the ball $B = \{y : |x - y| < \delta\}$ is contained in U_α. Then $B \subseteq U$. Hence U is open.

By taking complements it follows that the intersection of any family of closed sets is closed.

12. The Cantor set E is expressed as $\cap_{n=1}^{\infty} E_n$, where each E_n is a finite union of closed intervals and hence is closed. So by problem 11 E is closed.

13. Fix $a \in \mathbb{R}$ and let $A = \{x : f(x) > a\}$. We have to show A is a Borel set. Now $x \in A$ iff either $x > a$ and x is irrational, or $x < -a$ and x is rational. So $A = \{(a, \infty) \cap \mathbb{Q}^c\} \cup \{(-\infty, -a) \cap \mathbb{Q}\}$. As the intervals are Borel sets and $\mathbb{Q}$ is Borel, so also is A.

14. If $a \geq 0$ then $\{x : k(x) > a\} = \{x : h(x) > a^{1/2}\} \cup \{x : h(x) < -a^{1/2}\}$ is a Borel set. And if $a < 0$ then $\{x : k(x) > a\} = \mathbb{R}$ so in all cases it is Borel. Hence k is a Borel function.

Now if f and g are Borel functions then $f + g$ and $f - g$ are Borel (lectures) and by the first part so are $(f + g)^2$ and $(f - g)^2$. Then finally so is $fg = \frac{1}{4}\{(f + g)^2 - (f - g)^2\}$.

15. Suppose $B_n \in \mathcal{A}$ and $B = \cup_n B_n$; we need to show that $B_n \in \mathcal{A}$. Let $A_n = B_n \cap (B_1 \cup \cdots B_{n-1})^c \in \mathcal{A}$. Then the sets $A_1, A_2, \cdots$ are disjoint and $B = \cup_n A_n \in \mathcal{A}$.

16. Suppose f is $\mathcal{C}$-measurable. For each $a \in \mathbb{R}$ let $E_a = \{x \in \mathbb{R} : f(x) > a\}$. Then for each a, either E_a is countable or E_a^c is countable. Now $\mathbb{R} = \cup_{n \in \mathbb{Z}} E_n$ and $\mathbb{R} = \cup_{n \in \mathbb{Z}} E_n^c$ so we cannot have E_a countable for all a or E_a^c countable for all a. Also E_a countable implies E_d countable for $d > a$. Hence there is $b \in \mathbb{R}$ such that E_a is countable for $a > b$ and E_a^c is countable for $a < b$. Then $E_b = \cup_n E_{b+n^{-1}}$ is countable and $\{x : f(x) < b\} = \cup_n E_{b-n^{-1}}^c$ is also countable. Hence $\{x : f(x) \neq b\}$ is countable, as required.

Conversely, if there is a countable set E and $b \in \mathbb{R}$ such that $f(x) = b$ for $x \in E^c$ then the set $E_a\{x : f(x) > a\}$ is countable if $a \geq b$, and has countable complement if $a < b$, so in either case $E_a \in \mathbb{C}$, and hence f is $\mathbb{C}$-measurable.

17. If E is a subset of $[0, \infty)$ let $E_s = E \cup \{-x : x \in E\}$. Then $E \to E_s$ gives a one-one correspondence between subsets of $[0, \infty)$ and symmetric subsets of $\mathbb{R}$. This correspondence preserves complements (w.r.t. $[0, \infty)$ and $\mathbb{R}$ respectively) and countable unions, so σ-algebras on $[0, \infty)$ correspond to σ-algebras of symmetric subsets of $\mathbb{R}$. Hence $\mathcal{A}$ corresponds to the σ-algebra on $[0, \infty)$ generated by sets of the form $\{x \in [0, \infty) : x > b\}$, and this is the algebra of Borel subsets of $[0, \infty)$. So $\mathcal{A}$ consists of all sets of the form E_s where E is a Borel subset of $[0, \infty)$, and these are precisely the symmetric Borel subsets of $\mathbb{R}$.

18. Let $A, B \in \mathcal{A}$ be disjoint. Then A^c and B^c cannot both be finite. If A and B are both finite then $A \cup B$ is also finite and $\mu(A \cup B) = 0 = \mu(A) + \mu(B)$. If A and B^c are finite then $(A \cup B)^c$ is finite, so $\mu(A \cup B) = 1 = 0 + 1 = \mu(A) + \mu(B)$. So in all cases $\mu(A \cup B) = \mu(A) + \mu(B)$ and so μ is additive.

Now let $A_n = \{n\}$ for $n \in \mathbb{N}$. Then $\cup_n A_n = \mathbb{N}$ and $\mu(\mathbb{N}) = 1$ but $\mu(A_n) = 0$ for each n so $\sum \mu(A_n) = 0$ and so μ is not countably additive.

19. (Note: question should say $\mu(\{r\}) = (|p|^3 + q^3)^{-1}$). From lectures we know that there is a measure μ with the given property (and say $\mu(0) = 0$ - the case $r = 0$ not being covered by $r = \frac{p}{q}$ with $p \neq 0$). We have to show finiteness. Now

$$\mu(\mathbb{Q}) = \sum_{r \in \mathbb{Q}} \mu(r) \leq \sum_{p \neq 0, p \in \mathbb{Z}, q \in \mathbb{N}} (|p|^3 + q^3)^{-1} = 2 \sum_{p,q \in \mathbb{N}} (p^3 + q^3)^{-1}$$

$$\leq 4 \sum_{p,q \in \mathbb{N}; p \leq q} (p^3 + q^3)^{-1} = 4 \sum_{q=1}^{\infty} \sum_{p=1}^{q} (p^3 + q^3)^{-1} \leq 4 \sum_{q=1}^{\infty} \sum_{p=1}^{q} q^{-3} = 4 \sum_{q=1}^{\infty} q^{-2} < \infty$$

as required.

20. We first show that $\mu(A) \geq 0$ for all $A \in \mathcal{C}$. If A is countable this follows from the definition. If A^c is countable then $\mu(A^c) \leq \sum_n a_n \leq b$ so $\mu(A) = b - \mu(A^c) \geq 0$.

Next we show countable additivity. Let $A_1, A_2, \cdots \in \mathcal{C}$ be disjoint and let A be there union. Then it is impossible for more than one of the A_k to have countable complement. This leaves two cases. First, if all A_k are countable, then $\mu(A) = \sum_{x_n \in A} a_n = \sum_k \sum_{x_n \in A_k} a_n = \sum_k \mu(A_k)$. Second if one of the A_k, say A_1, has countable complement (and the rest are countable) then A^c is countable and $A_1^c = A^c \cup \cup_{k=1}^{\infty} A_k$. Thus $\mu(A_1^c) = \mu(A^c) + \sum_{k=2}^{\infty} \mu(A_k)$. hence $\mu(A) = b - \mu(A_c) = b - \mu(A_1^c) + \sum_{k=2}^{\infty} \mu(A_k) = \mu(A_1) + \sum_{k=2}^{\infty} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k)$ as required. Finally $\mu(\mathbb{R}) = b - \mu(\emptyset) = b < \infty$ so μ is a finite measure.

21. If $A \in \mathcal{G}$ then for some n, A depends only on the first n tosses, which means $A = \cup_{(s_1 \cdots s_n) \in E} A(s_1, \cdots, s_n)$ where E is a subset of the set of possible choices for $(s_1 \cdots s_n)$. Then we define $\mu(A) = 2^{-n} \#E$ (where $\#E$ is the number of members of E). This definition is independent of the choice of n - if we replace n by $n+1$, then each $(s_1 \cdots s_n) \in E$ is replaced by two sets $(s_1 \cdots s_n H)$ and $(s_1 \cdots s_n T)$, so the size of E doubles but we replace 2^{-n} by 2^{-n-1} so we get the same value for $\mu(A)$.

If now $A, B \in \mathcal{G}$ are disjoint, then we can choose n so that both depend only on the first n tosses, and then we can write $A = \cup_{(s_1 \cdots s_n) \in E} A(s_1, \cdots, s_n)$ and $B = \cup_{(s_1 \cdots s_n) \in F} A(s_1, \cdots, s_n)$, where E and F are disjoint. Then $\mu(A \cup B) = 2^{-n} \#(E \cup F) = 2^{-n} \#E + 2^{-n} \#F = \mu(A) + \mu(B)$ so μ is additive.

Now suppose $A(s_1, \cdots, s_n) \subseteq \cup_k E_k$ where $E_k \in \mathcal{G}$. Write $F_m = \cup_{k=1}^m E_m$. Suppose there is no m such that $A(s_1, \cdots, s_n) \subseteq F_m$. Then we can write $A(s_1, \cdots, s_n) = A(s_1, \cdots, s_n, H) \cup A(s_1, \cdots, s_n, T)$. Now, if $A(s_1, \cdots, s_n, H) \subseteq F_{m_1}$ and $A(s_1, \cdots, s_n, T) \subseteq F_{m_2}$ then we would have $A(s_1, \cdots, s_n) \subseteq F_m$ where $m = \max(m_1, m_2)$, and this is not true. So for at least one choice of $s_{n+1} = H$ or T we have that $A(s_1, \cdots, s_{n+1})$ is not contained in F_m for any m.

Then we can repeat this process and obtain $s_{n+2}, \cdots$ so that, for any $r \geq n$, $A(s_1, \cdots, s_r)$ is not contained in any F_m. Then let $\omega = (s_1, s_2, \cdots)$. Then $\omega \in A(s_1, \cdots, s_n) \subseteq \cup E_k$ so $\omega \in E_m$ for some m. But $E_m \in \mathcal{G}$, and so depends only on the first r tosses, for some r. But then $A(s_1, \cdots, s_r) \subseteq E_m \subseteq F_m$, a contradiction. This shows that $A(s_1, \cdots, s_n) \subseteq F_m$ for some m as required.

Now we use this result to show μ is countably additive on $\mathcal{G}$. Suppose $A \in \mathcal{G}$ and $E_1, E_2, \cdots \in \mathcal{G}$ are disjoint with union A. We have to show $\mu(A) = \sum \mu(E_k)$. As $A \in \mathcal{G}$ we can write $A = \cup_{j=1}^{r} B_j$ where each B_j is of the form $A(s_1, \cdots, s_n)$, so that by the above result $B_j \subseteq \cup_{k=1}^{m_j}$. Then $A \subseteq \cup_{k=1}^{m} E_k$ where $m = \max_{j=1}^{r} m_j$. But the E_k are disjoint and $A = \cup_{k=1}^{\infty} E_k$. Hence $A = \cup_{k=1}^{m} E_k$ and $E_k = \emptyset$ for $k > m$. So $\mu(A) = \sum_{k=1}^{m} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k)$, proving countable additivity.

22. We use the correspondence between E and the set Ω of problem 21, as described in section 1.4. This is a one-one correspondence under which the σ-algebra $\mathcal{B}_E$ of Borel subsets of E maps to the σ-algebra $\mathcal{F}$ on Ω generated by $\mathcal{G}$. Then the measure μ on $\mathcal{F}$ constructed in problem 19 corresponds to a measure ν on $\mathcal{B}_E$ - meaning that if $B \in \mathcal{B}_E$ then $\nu(B) = \mu(A)$ where A is the corresponding set in $\mathcal{F}$. Then if I is one of the constituent intervals of E_n, $I \cap E$ corresponds to $A(s_1, \cdots, s_n)$ for a suitable choice of $s_1 \cdots s_n$, so that $\nu(I \cap E) = 2^{-n}$.

23. First note that if $\epsilon > 0$ then we can find $F_1, F_2, \cdots \in \mathcal{A}$ such that $\sum \mu(F_k) \leq \lambda(A) + \epsilon$, and $A \subseteq \cup F_k$. Then if we let $F = \cup F_k$ we have $F \in \mathcal{F}$, $A \subseteq F$, and $\lambda(F) \leq \lambda(A) + \epsilon$. Using this with $\epsilon = n^{-1}$, we can find, for each $n \in \mathbb{N}$, a set $E_n \in \mathcal{F}$, such that $A \subseteq E_n$ and $\lambda(E_n) \leq \lambda(A) + n^{-1}$. Then let $= \cap_n E_n \in \mathcal{F}$. Then $A \subseteq E$, so $\lambda(E) \geq \lambda(A)$. Also $\lambda(E) \leq \lambda(E_n) \leq \lambda(A) + n^{-1}$ for each n, so $\lambda(E) \leq \lambda(A)$. Hence $\lambda(E) = \lambda(A)$.

24. For $n \in \mathbb{N}$ let $F_n = E_n \cap E_{n+1}^c$. Then $F_n \in \mathcal{F}$, the sets $F_1, F_2, \cdots$ and E are all disjoint, and $E_n = E \cup (\cup_{k=n}^{\infty} F_k)$ for each $n \in \mathbb{N}$. Then $\mu(E_n) = \mu(E) + \sum_{k=n}^{\infty} \mu(F_n)$. So the series $\sum \mu(F_n)$ converges, and $\mu(E_n) - \mu(E) = \sum_{k=n}^{\infty} \mu(F_n) \to 0$ as $n \to \infty$.

25. $\mu(E_n) = \sum_{k \in E_n} \mu(\{k\}) = \infty$ since there are infinitely many $k \in E_n$, and $\mu(\{k\}) = 1$ for each. If $m \in \cap_n E_n$ then $m \geq n$ for all n, which is impossible, so $\cap_n E_n = \emptyset$, so $\mu(\cap_n E_n) = 0$.

26. Let E be the set of $x \in [0, 1]$ having no 7 in its decimal expansion. Then $E = \cap_n E_n$ where E_n is the set of $x \in [0, 1$ having no 7 in the first n places. Then E_n is the union of 9^n intervals of length 10^{-n} and so $\lambda(E_n) = (\frac{9}{10})^n$ and so $\lambda(E) = 0$. Likewise for each $k \in \mathbb{Z}$ the set of $x \in [k, k+1]$ having no 7 following the decimal point has measure 0, and taking union over k the set of $x \in \mathbb{R}$ having no 7 has measure 0.

Now let F_r be the set of $x \in \mathbb{R}$ having no 7 in its decimal expansion following the r'th place after the decimal point. Then in the same way as above we have $\lambda(F_r) = 0$, and so $\lambda(F) = 0$ where $F = \cup_{r=1}^{\infty} F_r$ is the set of $x \in \mathbb{R}$ having at most finitely many 7's in its decimal expansion.

For the second part, we consider the decimal expansion of x in blocks of 10 digits following the decimal point. Let now E_n be the set of $x \in [0, 1]$ such that none of the first n blocks is precisely 9876543210. E_n is a union of $(10^{10} - 1)^n$ intervals of length 10^{-10n} so $\lambda(E_n) = (1 - 10^{-10})^n \to 0$ as $n \to \infty$. Then the set of $x \in [0, 1]$ not having the sequence 9876543210 anywhere in its expansion is a subset of $\cap_n E_n$, and hence has measure 0. In the same way as before it follows that the set of $x \in \mathbb{R}$ having at most finitely many occurrences of this sequence has measure 0.

Finally let Λ be the set of all finite sequences of digits from $\{0, 1, \cdots, 9\}$. Then Λ is countable (this is essentially problem 6(iii)). For each sequence $\sigma \in \Lambda$ let G_σ be the set of $x \in \mathbb{R}$ which have at most finitely many occurrences of the sequence σ in their expansion. The argument given above for 9876543210 can be applied to any sequence and gives $\lambda(G_\sigma) = 0$. Let $G = \cup_{\sigma \in \Lambda} G_\sigma$. Then (by the countability of Λ) we have $\lambda(G) = 0$, and if $x \in G^c$ then every sequence σ occurs infinitely often in the expansion of x.

27. We have $\phi(s) = \frac{1}{2}(s^{-\alpha} + s^{1-\alpha})$ so $\phi'(s) = \frac{1}{2}(-\alpha s^{-1-\alpha} + (1-\alpha)s^{-\alpha})$ which is 0 for $s = s_0 = \alpha(1-\alpha)^{-1}$. We see also that $\phi'(s) > 0$ for $s > s_0$ and < 0 for $s < s_0$, so the minimum value of ϕ is $\phi(s_0) = \frac{1}{2}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}$ and then $\mathbb{P}(X_n \leq \alpha n) \leq \phi(s_0)^n$ as in the notes.

28. Let $\mathcal{G}_n$ be the algebra of sets which depend only on the first n tosses. Each set in $\mathcal{G}_n$ is a union of sets of the form $A(s_1, \cdots, s_n)$, and we can define an additive set function $\mathbb{P}_n$ on $\mathcal{G}_n$ by setting $\mathbb{P}_n(A(s_1, \cdots, s_n)) = p^k q^{n-k}$ as in the question. To obtain a set function on $\mathcal{G}$ we need to check consistency, i.e. that $\mathbb{P}_n$ and $\mathbb{P}_{n+1}$ agree on $\mathcal{G}_n$. For this it is enough to check it for $A(s_1, \cdots, s_n)$. If $s_1, \cdots, s_n$ consists of k H's and $n-k$ T's then, since $A(s_1, \cdots, s_n) = A(s_1, \cdots, s_n, H) \cup A(s_1, \cdots, s_n, T)$, a disjoint union, we have $\mathbb{P}_{n+1}(A(s_1, \cdots, s_n)) = \mathbb{P}_{n+1}(A(s_1, \cdots, s_n, H)) + \mathbb{P}_{n+1}(A(s_1, \cdots, s_n, T)) = p^{k+1}q^{n-k} + p^k q^{n-k+1} = p^k q^{n-k} = \mathbb{P}_n(A(s_1, \cdots, s_n))$ as required.

Thus we obtain an additive set function $\mathbb{P}$ on $\mathcal{G}$. It now follows from the argument used in problem 21 that it is countable additive (that argument in fact shows that *any* finitely additive set function on $\mathcal{G}$ is countably additive).

29. We have, for any s with $0 < s \le 1$, that

$$\mathbb{P}(X_n \le \alpha n) = \sum_{k \le \alpha n} \binom{n}{k} p^k q^{n-k} \le s^{-\alpha n} \sum_{k \le \alpha n} \binom{n}{k} s^k p^k q^{n-k} \le s^{-\alpha n}(ps+q)^n = \phi(s)^n$$

where $\phi(s) = s^{-\alpha}(ps+q)$. To get the most from this inequality we choose $s \in (0,1]$ to minimise $\phi(s)$. We find that $\phi'(s) = s^{-\alpha-1}\{p(1-\alpha)s - q\alpha\}$. So if we set $s_0 = \frac{q\alpha}{p(1-\alpha)}$, and note that $s_0 < 1$ since $\alpha < p$ and $q = 1-p < 1-\alpha$, then we have $\phi'(s_0) = 0$, and also $\phi'(s)$ is positive for $s_0 < s \le 1$ and negative for $0 < s < s_0$, so we can conclude that ϕ has a minimum at s_0 and $\phi(s_0) < \phi(1) = 1$. So we set $u = \phi(s_0) = (\frac{q}{1-\alpha})^{1-\alpha}(\frac{p}{\alpha})^\alpha$ and then $u < 1$ and $\mathbb{P}(X \le \alpha n) \le u^n$.

For the second part, we follow the argument in section 2.3 and let $B_{nk} = \{\omega : X_n \le n(p - \frac{1}{k})\}$ and $C_{nk} = \{\omega : X_n \ge n(p + \frac{1}{k})\}$, and we need to show that $\sum_n \mathbb{P}(B_{nk}) < \infty$ and $\sum_n \mathbb{P}(C_{nk}) < \infty$ for each k. So we fix k and apply the first part with $\alpha = p - \frac{1}{k}$, using the fact that X_n has $B(n,p)$ distribution, and obtain $u < 1$ such that $\mathbb{P}(B_{nk}) \le u^n$ for each n. For C_{nk}, we use the fact that $C_{nk} = \{\omega : n - X_n < n(q - \frac{1}{k})\}$, and note that $n - X_n$ has $B(n,q)$ distribution. Then we can apply the first part with q in place of p and $\alpha' = q - \frac{1}{k} < q$, and obtain $u' < 1$ so that $\mathbb{P}(C_{nk}) \le u'^n$ for each n, and the rest of the argument goes as in the $p = \frac{1}{2}$ case.

30. We take the measure $\mathbb{P}$ of problem 25 and transfer it to E using the correspondence between Ω and E described in section 1.4. Then the set of points of E whose ternary expansion starts with a given sequence of 2's and 0's of length n corresponds to a set $A(s_1, \cdots, s_n)$, so the measure of the set is $p^k q^{n-k}$.

31. Define $\mu(E) = \lambda(\phi(E))$ for Borel sets E. This is well-defined since $\phi(E)$ is then a Borel set, and one easily checks μ is a measure, and $\mu([a,b]) = \lambda([\phi(a), \phi(b)]) = \phi(b) - \phi(a)$.

32. (i) By definition $\lambda_2(E)$ is $\inf \sum_n \lambda_2(R_n)$ taken of all sequences of rectangles R_n of the form $[a,b) \times [c,d)$ such that $E \subseteq \cup_n R_n$. Now, given such a sequence R_n we can find, for each n, a sequence (Q_{nm}) of non-overlapping closed squares with sides parallel to the axes, such that $R_n = \cup_k R_{nm}$, and then $\lambda_2(R_n) = \sum_m \lambda_2(Q_{nk})$. Then $E \subseteq \cup_{n,m} Q_{nm}$ and $\sum_{n,m} \lambda_2(Q_{nm}) = \sum_n \lambda_2(R_n)$. It follows that $\lambda_2(E) \ge \inf \sum_k \lambda_2(Q_k)$ and the reverse inequality follows from the countable additivity of λ_2.
(ii) For a square Q with sides parallel to the axes the result is immediate since $\lambda_2(Q)$ is l^2 where l is the length of its side. Now if $E \in \mathcal{L}^2$ and $\tilde{E} = \{ax : x \in E\}$ then, if (Q_k) is a sequence of axis-parallel squares with $E \subseteq \cup_k Q_k$ then $\tilde{E} \subseteq \cup_k \tilde{Q}_k$ where $\tilde{Q}_k = \{ax : x \in Q_k\}$ and then $\lambda(\tilde{E}) \le \sum_k \lambda_2(\tilde{E}_k) = a^2 \sum_k \lambda_2(E_k)$ and it follows that $\lambda(\tilde{E}) \le a^2 \lambda_2(E)$ and the reverse inequality in proved in the same way. The proof for $\{x + c : x \in E\}$ is similar.
(iii) Suppose Q' is an axis-parallel square with side b. Then $\lambda_2(Q') = b^2$. And $R_\theta(Q')$ is obtained from $R_\theta(Q)$ by translation and multiplication by b, so by (ii) $\lambda_2(Q') = b^2\lambda_2(Q) = b^2 a = a\lambda_2(Q')$. It

the follows from (i), by the same sort of argument as used in (ii), that $\lambda_2(R_\theta(E)) = a\lambda_2(E)$ for any $E \in \mathcal{L}^2$.

Then $\lambda_2(R_{2\theta}(Q)) = \lambda_2(R_\theta(R_\theta(Q))) = a^2$ and repeating we get $\lambda_2(R_{n\theta}(Q)) = a^n$. Now let $Q_- = \{2^{-1/2}x : x \in Q\}$ and $Q_+ = \{2^{1/2}x : x \in Q\}$. Then $\lambda_2(Q_-) = \frac{1}{2}$, and $\lambda_2(Q_+) = 2$, and $Q_- \subseteq R_{n\theta}(Q) \subseteq Q_+$ for each n. Hence $\frac{1}{2} \le a^n \le 2$ for all $n \in \mathbb{N}$ so $a = 1$. Hence $\lambda_2(R_\theta(E)) = \lambda_2(E)$ for all $E \in \mathcal{L}^2$.

33. Let $X_n = \sum_{k=1}^n 2^{-k}Y_k$, then X_n is simple, taking the values $j2^{-n}$ for $j = 0, 1, \cdots, 2^n - 1$, the sequence (X_n) is increasing and $X_n \to X$ pointwise. And $\mathbb{E}X_n = \sum_{k=1}^n 2^{-k}\mathbb{E}Y_k = p(1 - 2^{-n})$ since $\mathbb{E}Y_k = p$ for each k. Hence $\mathbb{E}X = \lim_n \mathbb{E}X_n = p$.

34. If $x \in [0, 1)$ has triadic expansion $0.\epsilon_1\epsilon_2\cdots$, where each ϵ_i is 0,1 or 2, then $x = \sum_{k=1}^\infty 3^{-k}\epsilon_k$. When $x \in E$, each ϵ_k is 0 or 2, and under the correspondence with the coin-tossing sample space we have $\epsilon_k = 2Y_k$, where Y_k is as in problem 33, and so $x = \sum_{k=1}^\infty 3^{-k}Y_k$. Then, since ν corresponds to the probability $\mathbb{P}$, we have $\int x d\nu(x) = \mathbb{E}(2\sum_{k=1}^\infty 3^{-k}Y_k) = 2p\sum_{k=1}^\infty 3^{-k} = p$ since $\mathbb{E}Y_k = p$.

35. For each $n \in \mathbb{Z}$ let $g(\omega) = n\epsilon$ for all ω such that $n\epsilon \le f(\omega) < (n + 1)\epsilon$. For each $\omega \in \Omega$ there is a unique n with this property, so g is well-defined, and it has the required properties.

36. For any choice of $Y_k = 0$ or 1 (i.e. for any ω), $Y(\omega)$ has a ternary expansion whose kth digit is $2Y_k$. So $Y(\omega) \in E$, and hence ρ_Y is a measure on E. Now if I is a component interval of E_n for some n, corresponding to a sequence of n digits each 0 or 2, then $Y^{-1}(I) = A(s_1, \cdots, s_n)$ where $(s_1 \cdots s_n)$ is the corresponding sequence of H's and T's. So $\rho_Y(I) = \mathbb{P}(A(s_1, \cdots, s_n)) = \nu(I)$ where ν is the measure found in problem 27. This is true for all component intervals of all E_n, so $\rho_Y(F) = \nu(F)$ for all sets in the σ-algebra generated by such I, i.e. for all Borel subsets of E. So $\rho_Y = \nu$.

37. $\int_0^1 x^q(\log x)^p dx = [\frac{x^{q+1}}{q+1}(\log x)^p]_0^1 - \int_0^1 \frac{x^{q+1}}{q+1}\frac{p}{x}(\log x)^{p-1}dx = -\frac{p}{q+1}\int_0^1 x^q(\log x)^{p-1}dx$. Repeating we get $\int_0^1 x^q(\log x)^p dx = \frac{(-1)^p p!}{(q+1)^p}\int_0^1 x^q dx = \frac{(-1)^p p!}{(q+1)^{p+1}}$.

Then $x^{-x} = e^{-x\log x} = \sum_{n=0}^\infty \frac{x^n(-\log x)^n}{n!}$. Now $\log x < 0$ for $0 < x < 1$ so each term in the series is positive, and we can apply the MCT to deduce $\int_0^1 x^{-x}dx = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^1 x^n(\log x)^n dx = \sum_{n=0}^\infty (n+1)^{-n-1} = \sum_{n=1}^\infty n^{-n}$.

38. If $x \ge 0$ then $\mathbb{P}(Y \le x) = \mathbb{P}(X^2 \le x) = \mathbb{P}(X \le x^{1/2}) = \min(x^{1/2}, 1)$. So $F_Y(x) = 0$ if $x < 0$, $x^{1/2}$ if $0 \le x \le 1$, and 1 if $X > 1$.

39. If $x > 0$ then $\mathbb{P}(Z > x) = \mathbb{P}(X > x$ and $Y = 1) = pe^{-x}$ so $F_Z(x) = 1 - pe^{-x}$. Similarly if $x < 0$ then $\mathbb{P}(X < x) = (1-p)e^x$. Then we see that the density function is $f_Z(x) = pe^{-x}$ for $x > 0$ and $(1-p)e^x$ for $x < 0$. Then $\mathbb{E}(e^{Z/2}) = \int_{-\infty}^\infty e^{x/2}f_Z(x)dx = (1-p)\int_{-\infty}^0 e^{3x/2}dx + p\int_0^\infty e^{-x/2}dx = \frac{2}{3}(1-p) + 2p = \frac{2}{3}(1+2p)$.

40. (a) If E is a Borel set let $E^+ = E \cap [0, \infty)$. Then $f^{-1}(E) = E^+ \cup \{x : -x \in E^+\}$ which is a symmetric Borel set, and any symmetric Borel set F is equal to $f^{-1}(F)$, so $\sigma(f)$ is the collection of all symmetric Borel sets in $\mathbb{R}$.
(b) $f^{-1}(\{x > \lambda\}) = \{x > \lambda^{1/3}\}$ and any $a \in \mathbb{R}$ is $\lambda^{1/3}$ for some λ so $\sigma(f)$ is the σ-algebra generated by all sets $\{x > a\}$ for $a \in \mathbb{R}$, i.e. $\mathcal{B}$.
(c) If F is a subset of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we define $\rho(F) = \cup_{n=0}^\infty(F_n \cup G_n)$ where $F_n = \{2n\pi + x : x \in F\}$ and $G_n = \{(2n+1)\pi - x : x \in F\}$. Then if $E \in \sigma(f)$, and $F = E \cap [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $E = \rho(F)$, and F is a Borel subset of $[-\frac{\pi}{2}, \frac{\pi}{2}]$. On the other hand, if F is a Borel subset of $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $L = \{\sin x : x \in F\}$, then L is a Borel set (since $\sin^{-1}$ is continuous on [0,1]), and $\rho(F) = f^{-1}(L)$ is in $\sigma(f)$. So $\sigma(F)$ is the collection of all sets of the form $\rho(F)$ where F is a Borel subset of $\{-\frac{\pi}{2}, \frac{\pi}{2}]$.

41. $\mathbb{P}(X_n > \alpha\log n) = e^{-\alpha\log n} = n^{-\alpha}$ and $\sum n^{-\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \le 1$ so the result follows from the Borel-Cantelli theorem.

$\mathbb{P}(Y_n < \alpha \log n) = \mathbb{P}(X_k < \alpha \log n \text{ for } k = 1, 2, \cdots, n) = (1 - n^{-\alpha})^n < (e^{-n^{-\alpha}})^n = e^{-n^{1-\alpha}}$ using the fact that $1 - x < e^{-x}$ for $x > 0$. Then if $\epsilon > 0$ and we set $\alpha = 1 - \epsilon$, we have $n^\epsilon > 2 \log n$ for large n so $e^{-n^\epsilon} < n^{-2}$ and then $\sum \mathbb{P}(\frac{Y_n}{\log n} < 1 - \epsilon) < \sum e^{-n^\epsilon} < \infty$ and by Borel-Cantelli again, a.s. $\frac{Y_n}{\log n} \geq 1 - \epsilon$ for n large enough. On the other hand, by the first part a.s. $\frac{Y_n}{\log n} < 1 + \epsilon$ for n large enough, so a.s. $\frac{Y_n}{\log n} \to 1$.

42. $\mathbb{P}(X_n \geq k) \leq \sum_{j=1}^n \mathbb{P}(\text{tosses number } j, j+1, \cdots, j+k-1 \text{ are all } H) \leq n2^{-k}$. Considering m disjoint blocks of k tosses, we have $\mathbb{P}(X_n < k) \leq \mathbb{P}(\text{none of the blocks is all } H) = (1 - 2^{-k})^m$. Now if k is the smallest integer $> (1 + \epsilon) \log_2 n$ then $\mathbb{P}(A_n) = \mathbb{P}(X_n \geq k) \leq n2^{-(1+\epsilon)\log_2 n} = n^{-\epsilon}$. So $\sum_r \mathbb{P}(A_{2^r}) \leq \sum_r 2^{-r\epsilon} < \infty$. Similarly $\mathbb{P}(B_n) \leq \{1 - 2^{-(1-\epsilon)\log_2 n}\}^{\frac{n}{(1-\epsilon)\log_2 n}} = (1 - n^{\epsilon-1})^{\frac{n}{(1-\epsilon)\log_2 n}} \leq e^{-n^{\epsilon/2}}$ for n large. So $\sum_n \mathbb{P}(B_n) < \infty$. Then from Borel-Cantelli we deduce that with probability 1, $1 - \epsilon < \frac{X_{2^r}}{r} < 1 + \epsilon$ for r large enough. Now (X_n) is increasing so if $2^r \leq n < 2^{r+1}$ then $\frac{X_{2^r}}{r+1} \leq \frac{X_n}{\log_2 n} \leq \frac{X_{2^{r+1}}}{r}$. Now $\frac{X_{2^r}}{r+1} = \frac{X_{2^r}}{r}(\frac{r}{r+1}) \to 1$ a.s. and the same for $\frac{X_{2^{r+1}}}{r}$, so $\frac{X_n}{\log_2 n} \to 1$ a.s.

43. For $n \geq 1$ we have $I_n = I_{n-1} + [\sin x \frac{(\cos x)^{2n-1}}{2n-1}]_0^{\pi 2} - \frac{1}{2n-1} \int_0^{\pi/2} \cos x (\cos x)^{2n-1} dx = I_{n-1} - \frac{1}{2n-1} I_n$ so $\frac{2n}{2n-1} I_n = I_{n-1}$. Now $(n+1)(2n-1)^2 = 4n^3 - 3n + 1 < 4n^3$ so $(\frac{2n-1}{2n})^2 < \frac{n}{n+1}$ for $n \geq 1$ and so $I_n < \sqrt{\frac{n}{n+1}} I_{n-1}$ and by iteration we get $I_n < \frac{1}{\sqrt{n+1}} I_0 = \frac{\pi}{2\sqrt{n+1}}$.

Next, putting $u = kx$ gives $\int_0^\pi (\cos kx)^{2k^3} dx = \frac{1}{k} \int_0^{k\pi} (\cos)^{2k^3} du = 2 \int_0^{\pi/2} (\cos u)^{2k^3} du$ since each of the $2k$ intervals of length $\pi/2$ comprising $[0, \pi]$ gives an equal contribution. So $\int_0^\pi (\cos kx)^{2k^3} dx = 2I_{k^3} < \frac{\pi}{\sqrt{k^3+1}}$. Now $\frac{\pi}{\sqrt{k^3+1}} \leq \pi k^{-3/2}$ so the series $\sum_{k=1}^\infty \frac{\pi}{\sqrt{k^3+1}}$ converges by comparison, and so by the MCT the series $\sum_{k=1}^\infty (\cos kx)^{2k^3}$ converges a.e. to a finite sum $f(x)$, and $\int_0^\pi f(x) dx$ is finite. Also f is periodic with period π so f is integrable on every bounded interval.

44. (i) On $[-1, 1]$, $f(x) \leq |x|^{-1/2}$, which is integrable on $[-1, 1]$, and on $|x| \geq 1$, $f(x) \leq |x|^{-3/2}$ which is integrable on $|x| \geq 1$. Hence f is integrable on $\mathbb{R}$.
(ii) $|f(x)| \leq \frac{1}{1+x^2}$ so f is integrable.
(iii) $f(x) \to 1$ as $x \to -\infty$ so $f \geq 1/2$ on an infinite interval and hence f is not integrable integrable.
(iv) $f(x) \leq e^{-2x}$ for x large and positive, and $f(x) \leq e^{x/2}$ for x large and negative. And f is continuous so is integrable on any bounded interval, hence f is integrable on $\mathbb{R}$.

(v) Writing $g(x) = \sum_{k=1}^\infty (\cos kx)^{2k^3}$ and $K = \int_0^\pi g$, which is finite by problem 43, we have $\int_{m\pi}^{(m+1)\pi} g = K$ by the periodicity of g. Then for $m \geq 0$, on $[m\pi, (m+1)\pi]$ we have $f(x) \leq \frac{g(x)}{1+m^2\pi^2}$ so $\int_{m\pi}^{(m+1)\pi} f \leq \frac{K}{1+m^2\pi^2}$. Hence $\int_0^\infty f = \sum_{m=0}^\infty \int_{m\pi}^{(m+1)\pi} f \leq K \sum_{m=0}^\infty \frac{1}{1+m^2\pi^2} < \infty$. Since f is even it s also integrable on $(-\infty, 0]$ and hence on $\mathbb{R}$.

45. For $0 \leq x < 1$, we have $(1-x^2)^{-1/2} = 1 + (-\frac{1}{2})(-x^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x^2)^2 + \cdots = \sum_{n=0}^\infty \frac{1.3\cdots(2n-1)}{n!2^n} x^{2n} = \sum_{n=0}^\infty \binom{2n}{n} 4^{-n} x^{2n}$. Since all the terms are nonnegative the MCT gives $\int_0^1 (1-x^2)^{-1/2} dx = \sum_{n=0}^\infty \binom{2n}{n} 4^{-n} \int_0^1 x^{2n} dx = \sum_{n=0}^\infty \frac{4^{-n}}{2n+1} \binom{2n}{n}$. The integral is $[\sin^{-1} x]_0^1 = \frac{\pi}{2}$.

46. $-\frac{\log(1-x)}{x} = \frac{1}{x}(x + \frac{x^2}{2} + \frac{x^2}{3} + \cdots)$ for $0 \leq x < 1$, and as all the terms are nonnegative the MCT gives $-\int_0^1 \frac{\log(1-x)}{x} = \sum_{n=1}^\infty \frac{1}{n} \int_0^1 x^{n-1} dx = \sum_{n=1}^\infty \frac{1}{n^2}$ (note sign error in original version of question).

47. We have $\int_0^n \frac{1}{x}\{1 - (1 - \frac{x}{n})^n\} dx = \int_0^1 \frac{1-u^n}{1-u} du = \int_0^1 (1 + u + \cdots + u^{n-1}) du = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and $\int_1^n \frac{1}{x} dx = \log n$ giving the first part. Now, writing $g_n(x) = \frac{1}{x}\{1 - (1 - \frac{x}{n})^n\}$ and $f_n(x) = \frac{1}{x}(1 - \frac{x}{n})^n \chi_{[1,n]}(x)$, we have $g_n(x) \to g(x) = \frac{1}{x}(1 - e^{-x})$ pointwise on $(0, 1]$ and $f_n(x) \to f(x) = \frac{e^{-x}}{x}$

pointwise on $[1,\infty)$. Since $(1-\frac{x}{n})^n \le e^{-x}$ we have $f_n(x) \le e^{-x}$ on $[1,\infty)$ and the DCT gives $\int_1^\infty f_n \to \int_1^\infty f$. For g_n, note first that if $\phi(u) = u^n - nu$ then $\phi'(u) = n(u^{n-1}-1) \le 0$ for $u \in [0,1]$, and with $u = 1-\frac{x}{n}$ this gives $(1-\frac{x}{n})^n \ge 1-x$. This gives $g_n(x) \le 1$, and we also have $g_n(x) \ge 0$, so we can apply DCT again and get $\int_0^1 g_n \to \int_0^1 g$. Putting everything together now gives $\gamma_n \to \int_0^1 g - \int_1^\infty f$ as $n \to \infty$.

48. Let $Y_k = \log X_k$. Then $\mathbb{E}Y_k = \int_0^1 \log x dx = -1$ and $\mathbb{E}(Y_k^2) = \int_0^1 (\log x)^2 dx < \infty$ so the SLLN gives a.s. $n^{-1}\sum_{k=1}^n Y_k \to -1$ so $(X_1 \cdots X_n)^{1/n} \to e^{-1}$.