1. **(a)** State what is meant by an *algebra* and a *σ-algebra* of subsets of a set Ω. Give an example of an algebra which is not a σ-algebra.

[8 marks]

(b) What is meant by a *Borel subset* of $\mathbb{R}$?

Let (f_n) be a sequence of continuous real-valued functions of $\mathbb{R}$, and let F be the set of $x \in \mathbb{R}$ such that $f_n(x)$ converges to a limit in $\mathbb{R}$ as $n \to \infty$. Show that F is a Borel set.

[9]

(c) What does it mean to say that a function $f : \Omega \to \mathbb{R}$ is *measurable* with respect to a σ-algebra $\mathcal{A}$ on Ω?

Show that if (f_n) is a sequence of functions on Ω which are measurable w.r.t $\mathcal{A}$ and f_n converges pointwise to f on Ω then f is also measurable w.r.t. $\mathcal{A}$.

[8]

2. **(a)** Define the notions of *additivity* and *countable additivity* of a set function on an algebra of subsets of a set Ω.

Let $\mathcal{A}$ be the algebra of all subsets of $\mathbb{N}$ which are either finite or have finite complement, and define μ on $\mathcal{A}$ by $\mu(E) = 0$ if E is finite and $\mu(E) = 1$ if E^c is finite. Show that μ is additive but not countable additive.

[8]

(b) Let E be the set of all $x \in [0, 1]$ such that the decimal expansion of x does not contain the digit 3. Explain why E can be expressed as $E = \cap_{n=1}^{\infty} E_n$ where E_n is the union of 9^n intervals, each of length 10^{-n}. Deduce that E has Lebesgue measure 0.

Hence show that, if F is the set of real x such that the decimal expansion of x contains the digit 3 only a finite number of times, then F has Lebesgue measure zero.

[11]

Indicate briefly how your argument could be modified to show that the set of real x whose decimal expansion contains the sequence 345 only a finite number of times has Lebesgue measure zero.

[6]

3. (a) Define the notion of *simple function* on a measure space $(\Omega, \mathcal{F}, \mu)$ and explain briefly (without proofs) how simple functions are used to define the integral $\int f d\mu$ of a nonnegative measurable function f.

[8]

(b) Consider an infinite sequence of tosses of a fair coin. Define a random variable Y by $Y = \sum_{k=1}^{\infty} 4^{-k} Y_k$ where $Y_k = 1$ if the kth toss is a head and 0 if it is a tail. Find an increasing sequence of simple functions converging to Y and hence evaluate $\mathbb{E}Y$.

[7]

(c) State the Monotone Convergence Theorem, and use it to show that

$$\int_0^1 x^{-2}\{x + \log(1-x)\}dx = -\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

By writing $\frac{1}{n(n+1)}$ as a difference of two fractions, or otherwise, evaluate the sum on the right.

[10]

4. (a) Let (E_k) be a sequence of events on a probability space $(\omega, \mathcal{F}, \mathbb{P})$. Suppose that $\sum_{k=1}^{\infty} \mathbb{P}(E_k) < \infty$. Show that, with probability 1, E_k occurs for only finitely many values of k.

Hence show that, if (X_k) is a sequence of random variables each having an exponential distribution with expectation 1 (i.e. $\mathbb{P}(X_k > x) = e^{-x}$ for $x \geq 0$), and if $\alpha > 1$, then with probability 1 there exists n such that $X_k \leq \alpha \log k$ for all $k > n$.

[8]

(b) State the Dominated Convergence Theorem, and apply it to the series $1 - x^3 + x^6 - \cdots$ to show that

$$\int_0^1 \frac{dx}{1+x^3} = 1 - \frac{1}{4} + \frac{1}{7} - \cdots$$

[9]

(c) State the Strong Law of Large Numbers.

Let $X_1, X_2, \cdots$ be independent random variables, each having a uniform distribution on $[0, 1]$. Let $Y_n = (X_1 \cdots X_n)^{1/n}$. Show that, with probability 1, $Y_n \to e^{-1}$ as $n \to \infty$.

[8]

[End of Paper]

Solutions

1. (a) A collection $\mathcal{A}$ of subsets of Ω is an algebra if (i) $\Omega \in \mathcal{A}$, (ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$ and (iii) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. For a σ-algebra (iii) is replaced by $A_1, A_2, \cdots \in \mathcal{A}$ implies $\cup a_n \in \mathcal{A}$.

One example of an algebra which is not a σ-algebra is the collection $\mathcal{A}$ of subsets of $\mathbb{N}$ which are either finite or have finite complement. Then the set of all even integers is not in $\mathcal{A}$ but is the union $\cup_{k=1}^{\infty}\{2k\}$ of a sequence of subsets of $\mathcal{A}$, so $\mathcal{A}$ is not a σ-algebra.

(b) A Borel set is a set belonging to the σ-algebra $\mathcal{B}$ generated by all intervals of the form $[a, b)$ with $a, b \in \mathbb{R}$.

Note that any open set in $\mathbb{R}$ is a countable union of intervals of this form, and hence is a Borel set.

Now F is the set of all $x \in \mathbb{R}$ such that, for all $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ we have $|f_m(x) - f_n(x)| < \frac{1}{k}$. Thus $F = \cap_{k=1}^{\infty} \cup_{n_0=1}^{\infty} \cap_{m>n_0} \cap_{n>n_0} E_{kmn}$ where $E_{kmn} = \{x : |f_m(x) - f_n(x)| < \frac{1}{k}$. Now E_{kmn} is open since $f_m - f_n$ is continuous, hence $E_{kmn} \in \mathcal{B}$, and as $\mathcal{B}$ is a σ-algebra it follows that $F \in \mathcal{B}$.

(c) f is measurable iff $\{\omega : f(\omega) > a\} \in \mathcal{A}$ for every $a \in \mathbb{R}$.

For any $a \in \mathbb{R}$ we have $\{x : f(x) > a\} = \{x$:for some $k \in \mathbb{N}$ there exists m such that for all $n \geq m$ we have $f_n(x) > a + k^{-1}\} = \cup_{k=1}^{\infty} \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} \{x : f_n(x) > a + k^{-1}\} \in \mathcal{A}$.

2. (a) An additive set function is a mapping $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(E \cup F) = \mu(E) + \mu(F)$ whenever E and F are disjoint sets in $\mathcal{A}$. μ is countably additive if in addition $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$ whenever $E_1, E_2, \cdots$ are disjoint sets in $\mathcal{A}$ with union $E \in \mathcal{A}$.

Let $A, B \in \mathcal{A}$ be disjoint. Then A^c and B^c cannot both be finite. If A and B are both finite then $A \cup B$ is also finite and $\mu(A \cup B) = 0 = \mu(A) + \mu(B)$. If A and B^c are finite then $(A \cup B)^c$ is finite, so $\mu(A \cup B) = 1 = 0 + 1 = \mu(A) + \mu(B)$. So in all cases $\mu(A \cup B) = \mu(A) + \mu(B)$ and so μ is additive.

Now let $A_n = \{n\}$ for $n \in \mathbb{N}$. Then $\cup_n A_n = \mathbb{N}$ and $\mu(\mathbb{N}) = 1$ but $\mu(A_n) = 0$ for each n so $\sum \mu(A_n) = 0$ and so μ is not countably additive.

(b) Let E be the set of $x \in [0, 1]$ having no 3 in its decimal expansion. Then $E = \cap_n E_n$ where E_n is the set of $x \in [0, 1$ having no 3 in the first n places. For each sequence $s_1 s_2 \cdots s_n$ with each $s_i \in \{0, 1, \cdots, 9\}$, the set of $x \in [0, 1]$ with decimal expansion starting $0.s_1 \cdots s_n$ is an interval of length 10^{-n}, and there are 9^n choices of $s_1 \cdots s_n$ with no $s_i = 3$. So E_n is the union of 9^n intervals of length 10^{-n}.

Then $\lambda(E_n) = (\frac{9}{10})^n$ and so $\lambda(E) \leq (\frac{9}{10})^n$ for each n, so $\lambda(E) = 0$.

Likewise for each $k \in \mathbb{Z}$ the set of $x \in [k, k+1]$ having no 3 following the decimal point has measure 0, and taking union over k the set of $x \in \mathbb{R}$ having no 3 has measure 0. Now let F_r be the set of $x \in \mathbb{R}$ having no 3 in its decimal expansion following the r'th place after the decimal point. Then in the same way as above we have $\lambda(F_r) = 0$, and so $\lambda(F) = 0$ where $F = \cup_{r=1}^{\infty} F_r$ is the set of $x \in \mathbb{R}$ having at most finitely many 3's in its decimal expansion.

For the second part, we consider the decimal expansion of x in blocks of 3 digits following the decimal point. Let now E_n be the set of $x \in [0, 1]$ such that none of the first n blocks is precisely 345. Then E_n is a union of 999^n intervals of length 1000^{-n} and the conclusion follows in the same way.

3. **(a)** A simple function is a measurable function which takes only finitely many values. A simple function f can be expressed as $f = \sum_{i=1}^{n} \lambda_i \chi_{E_i}$ where $E_i \in \mathcal{F}$. Then one defines $\int f d\mu = \sum_{i=1}^{n} \lambda_i \mu(E_i)$ and shows that this is independent of the choice of representation. One also shows that $f \leq g$ implies $\int f d\mu \leq \int g d\mu$ for simple f and g. Then if f is nonnegative and measurable, one constructs an increasing sequence (f_n) of simple functions such that $f_n \to f$ pointwise. Then the sequence $(\int f_n d\mu)$ is increasing, so tends to a limit, which is defined to be $\int f d\mu$. To show it is well-defined one checks that any other increasing sequence of simple functions converging to f gives the same result.

(b) Let $X_n = \sum_{k=1}^{n} 4^{-k} Y_k$, then X_n is simple, taking values $j4^{-n}$ for $j \in \{0, 1, \cdots, 4^n - 1\}$, the sequence (X_n) is increasing and $X_n \to X$ pointwise. And $\mathbb{E}X_n = \sum_{k=1}^{n} 4^{-k} \mathbb{E}Y_k = \frac{1}{2} \sum_{k=1}^{n} 4^{-k} = \frac{1}{6}(1 - 4^{-n})$ since $\mathbb{E}Y_k = \frac{1}{2}$ for each k. Hence $\mathbb{E}X = \lim_n \mathbb{E}X_n = \frac{1}{6}$.

(c) The MCT states that if (f_n) is an increasing sequence of nonnegative measurable functions and $f = \lim_n f_n$ then $\int f_n d\mu \to \int f d\mu$.

For $|x| < 1$ we have $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ so

$$-x^{-2}\{x + \log(1-x)\} = \frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n+1}$$

Since the terms in this series are all nonnegative, we can apply the MCT to the sequence of partial sums and deduce that $-\int_0^1 x^{-2}\{x + \log(1-x)\}dx = \sum_{n=1}^{\infty} \frac{1}{n+1} \int_0^1 x^{n-1} dx = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

We have $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ so the sum to N terms is $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots \frac{1}{N} - \frac{1}{N+1} = 1 - \frac{1}{N+1}$ and letting $N \to \infty$ gives $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

4. **(a)** If F is the event that E_k occurs for infinitely many k, then $F = \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} E_k$ and then for each m we have $\mathbb{P}(F) \leq \mathbb{P}(\cup_{k=m}^{\infty} E_k) \leq \sum_{k=m}^{\infty} \mathbb{P}(E_k)$. The assumption $\sum \mathbb{P}(E_k) < \infty$ implies that $\sum_{k=m}^{\infty} \mathbb{P}(E_k) \to 0$ as $m \to \infty$, so $\mathbb{P}(F) = 0$ as required.

$\mathbb{P}(X_k > \alpha \log k) = e^{-\alpha \log k} = k^{-\alpha}$ and $\sum k^{-\alpha}$ converges since $\alpha > 1$ so the result follows from the first part.

(b) The DCT states that if (f_n) is a sequence of measurable functions on Ω such that $f_n \to f$ a.e., and if there is a nonnegative function $\phi \in L^1(\mu)$ such that for each n, $|f_n| \leq \phi$ a.e., then $\int f_n d\mu \to \int f d\mu$.

Let $f_n(x) = 1 - x^3 + x^6 - \cdots + (-1)^n x^{3n} = \frac{1-(-x^3)^{n+1}}{1+x^3}$. Then $f_n(x) \to \frac{1}{1+x^3}$ pointwise on $[0,1)$ and $|f_n(x)| \leq 2$ on $[0,1)$ so the DCT, applied to the sequence (f_n) and Lebesgue measure on $[0,1]$, and with $\phi(x) = 2$, gives $\int_0^1 f_n \to \int_0^1 \frac{dx}{1+x^3}$. Since $\int_0^1 f_n = 1 - \frac{1}{4} + \frac{1}{7} - \cdots + \frac{(-1)^n}{3n+1}$ it follows that $\int_0^1 \frac{dx}{1+x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$.

(c) The SLLN states that if $X_1, X_2, \cdots$ are independent random variables with the same distribution, having expectation μ and finite variance, then with probability 1, $\frac{1}{n} \sum_{k=1}^{n} X_k \to \mu$ as $n \to \infty$.

Let $Z_k = \log X_k$. Then Z_k are independent with the same distribution, and $\mathbb{E}(Z_k) = \int_0^1 \log x dx = [x \log x - 1]_0^1 = -1$. Also $\mathbb{E}(Z_k^2) = \int_0^1 (\log x)^2 dx$ is finite so the variance is finite. Then by the SLLN, with probability 1, $\frac{1}{n} \sum_{k=1}^{n} Z_k \to -1$ and so $Y_n = \exp(\frac{1}{n} \sum_{k=1}^{n} Z_k) \to e^{-1}$.