Proof of Carathéodory extension theorem (Non-examinable)

We suppose we have an algebra $\mathcal{A}$ of subsets of Ω and a countable additive set function μ on $\mathcal{A}$. Associated with μ we have an *outer measure* which is a set function λ defined for *all* subsets of Ω as follows:

$$\lambda(E) = \inf \left\{ \sum_{k=1}^{\infty} \mu(A_k) : \ A_k \in \mathcal{A} \ \& \ E \subseteq \cup_{k=1}^{\infty} A_k \right\}$$

The following proposition summarises the properties of outer measure that we need:

Proposition. (i) for any sequence $E_1, E_2, \cdots$ of sets we have

$$\lambda\left(\cup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \lambda(E_k)$$

(ii) if $A \in \mathcal{A}$ then $\lambda(A) = \mu(A)$
(iii) if $A \in \mathcal{A}$ and E is any set then $\lambda(A \cap E) + \lambda(A^c \cap E) = \lambda(E)$

Proof. (i) Let $\epsilon > 0$. Then for each k we can find a sequence $A_{k1}, A_{k2}, \cdots$ of sets in $\mathcal{A}$ which cover E_k and such that $\sum_{j=1}^{\infty} \mu(A_{kj}) < \lambda(E_k) + \epsilon 2^{-k}$. Then the countable family (A_{kj}) covers the union $\cup E_k$ and $\sum_{k,j} \mu(A_{kj}) \leq \sum_k \{\lambda(E_k) + \epsilon 2^{-k}\} = \epsilon + \sum_k \lambda(E_k)$. As ϵ is arbitrary it follows that $\lambda(\cup E_k) \leq \sum_k \lambda(E_k)$ as required.
(ii) That $\lambda(A) \leq \mu(A)$ follows from the definition of λ (one can take $A_1 = A$ and the other A_k empty). For the converse suppose $A \subseteq \cup A_k$ where $A_k \in \mathcal{A}$. We have to show $\sum \mu(A_k) \geq \mu(A)$. To do this let $B_k = A_k \cap (A_1 \cup \cdots \cup A_{k-1})^c$ so that $A \subseteq \cup B_k$ and the B_k are disjoint. Then by the countable additivity of μ we have $\mu(A) \leq \sum \mu(B_k) \leq \sum \mu(A_k)$ as required.
(iii) From (i) we have $\lambda(A \cap E) + \lambda(A^c \cap E) \geq \lambda(E)$. To show the reverse inequality, let $\epsilon > 0$ and find $A_1, A_2, \cdots$ in $\mathcal{A}$ such that $E \subseteq \cup A_k$ and $\sum \mu(A_k) < \lambda(E) + \epsilon$. Then the sets $A_k \cap A$ cover $E \cap A$, so $\sum \mu(A_k \cap A) \geq \lambda(E \cap A)$ and similarly $\sum \mu(A_k \cap A^c) \geq \lambda(E \cap A^c)$ and since $\mu(A_k \cap A) + \mu(A_k \cap A^c) = \mu(A_k)$ we deduce that $\sum \mu(A_k) \geq \lambda(A \cap E) + \lambda(A^c \cap E)$. Hence $\lambda(E) + \epsilon \geq \lambda(A \cap E) + \lambda(A^c \cap E)$ and as ϵ is arbitrary $\lambda(E) \geq \lambda(A \cap E) + \lambda(A^c \cap E)$ as required.

Part (i) of the proposition says that λ is *countably subadditive*. In general it will not be additive on arbitrary sets. However as we shall see it is countably additive on a suitable collection of sets, and this is how we obtain our measure. We now state the main result:

Theorem. (Carathéodory extension theorem) Let $\mathcal{A}$ be an algebra of subsets of Ω and let $\mathcal{F}$ be the σ-algebra generated by $\mathcal{A}$. Let μ be a σ-finite countably additive set function on $\mathcal{A}$.

Then there is a unique measure λ on $\mathcal{F}$ such that $\lambda(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof. We define $\lambda(E)$, for arbitrary E, as the outer measure as introduced above. We shall show that it is in fact a measure on $\mathcal{F}$. To do this we introduce a collection of sets $\mathcal{C}$, consisting of all sets B such that

$$\lambda(B \cap E) + \lambda(B^c \cap E) = \lambda(E) \tag{1}$$

for every set E. Note first that by part (ii) of the proposition this equation is satisfied by every $B \in \mathcal{A}$, so $\mathcal{A} \subseteq \mathcal{C}$.

Next we show that $\mathcal{C}$ is an algebra. It is clear from (1) that $B \in \mathcal{C}$ implies $B^c \in \mathcal{C}$. So we suppose $A, B \in \mathcal{C}$ and aim to prove $A \cup B \in \mathcal{C}$. To do this, first note that as $A \in \mathcal{C}$ we get from (1) applied with $B^c \cap E$ in place of E that

$$\lambda(A^c \cap B^c \cap E) + \lambda(A \cap B^c \cap E) = \lambda(B^c \cap E) \tag{2}$$

and as $B \in \mathcal{C}$, applying (1) with $(A \cup B) \cap E$ in place of E gives

$$\lambda((A \cup B) \cap E) = \lambda(A \cap B^c \cap E) + \lambda(B \cap E) \tag{3}$$

Combining (2) and (3) gives

$$\lambda((A \cup B)^c \cap E) + \lambda((A \cup B) \cap E) = \lambda(B^c \cap E) + \lambda(B \cap E) = \lambda(E)$$

showing that $\mathcal{C}$ is an algebra.

Note that (3) shows that, if A and B are disjoints sets in $\mathcal{C}$, then $\lambda((A \cup B) \cap E) = \lambda(A \cap E) + \lambda(B \cap E)$. By iterating this we obtain, for any disjoint $B_1, \cdots, B_n \in \mathcal{C}$ and any set E, that

$$\lambda((B_1 \cup \cdots \cup B_n) \cap E) = \sum_{k=1}^{n} \lambda(B_k \cap E) \tag{4}$$

Now we finish the proof by showing that $\mathcal{C}$ is a σ-algebra and λ is countably additive on $\mathcal{C}$. To do this suppose $B_1, B_2, \cdots$ are disjoint sets in $\mathcal{C}$. Write $A_n = \cup_{k=1}^{n} B_k$ and $A = \cup_{k=1}^{\infty} B_k$. Now for any set E, by (4) we have $\lambda(A_n \cap E) = \sum_{k=1}^{n} \lambda(B_k \cap E)$.
On the other hand by part (i) of the proposition $\lambda(A \cap E) \leq \sum_{k=1}^{\infty} \lambda(B_k \cap E)$. Combining these we deduce $\lambda(A \cap E) = \sum_{k=1}^{\infty} \lambda(B_k \cap E)$ and $\lambda(A_n \cap E) \to \lambda(A \cap E)$. Now as $A_n \in \mathcal{C}$ we have $\lambda(A_n \cap E) + \lambda(A_n^c \cap E) = \lambda(E)$ so $\lambda(A_n \cap E) + \lambda(A^c \cap E) \leq \lambda(E)$ and letting $n \to \infty$ gives $\lambda(A \cap E) + \lambda(A^c \cap E) \leq \lambda(E)$. The reverse inequality follows from (i) of the proposition so we have (1) holding for A and $A \in \mathcal{C}$. Also taking $E = \Omega$ we have $\lambda(A) = \sum_{k=1}^{\infty} \lambda(B_k)$.

Hence we conclude that $\mathcal{C}$ is a σ-algebra containing $\mathcal{A}$ and that λ is a measure on it. Then $\mathcal{C}$ contains $\mathcal{F}$ so λ is a measure on $\mathcal{F}$.

To prove uniqueness suppose ν is a measure on $\mathcal{F}$ such that $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$. First note that $\nu(E) \leq \lambda(E)$ for any $E \in \mathcal{F}$, since if $A_1, A_2, \cdots \in \mathcal{A}$ are such that $E \subseteq \cup A_k$ then $\nu(E) \leq \sum \nu(A_k) = \sum \mu(A_k)$ using the countable additivity of ν. Next, if $E \in \mathcal{F}$ and $A \in \mathcal{A}$ with $\mu(A) < \infty$ we have $\nu(E \cap A) = \lambda(E \cap A)$. To see this, suppose on the contrary that $\nu(E \cap A) < \lambda(E \cap A)$. Then

$$\mu(A) = \nu(A) = \nu(E \cap A) + \nu(E^c \cap A) < \lambda(E \cap A) + \lambda(E^c \cap A) = \lambda(A) = \mu(A)$$

which is a contradiction. So $\nu(E \cap A) = \lambda(E \cap A)$ when $\mu(A) > 0$, as claimed.

We apply this to get $\nu(E \cap B_k) = \lambda(E \cap B_k)$ for each k and any $E \in \mathcal{F}$, and we deduce $\nu(E) = \sum \nu(E \cap B_k) = \sum \lambda(E \cap B_k) = \lambda(E)$, so $\nu = \lambda$ as required.