Notes on the BMV conjecture

1. First Version

The Bessis-Moussa-Villani conjecture [1], which has its origins in Mathematical Physics, states the following:

BMV conjecture, Version 1. Let d be a positive integer and let A and B be Hermitian $d \times d$ matrices. Define $f_{A,B}(z) = \mathrm{tr}\{\exp(A + zB)\}$ for $z \in \mathbb{C}$. Then there is a positive measure μ on $\mathbb{R}$ such that

$$f_{A,B}(z) = \int e^{zt} d\mu(t) \tag{1}$$

for all $z \in \mathbb{C}$.

The issue here is the *positivity* of the measure μ. It is not too hard to show the existence of a unique measure μ satisfying (1), and in fact μ is supported on $[\mu, \lambda]$ where μ and λ are the least and greatest eigenvalues of B. This is done using the asymptotic belaviour of $f = f_{A,B}$. f is clearly an entire function and one can check that $|f(z)| \leq Ce^{\lambda x}$ for $x \geq 0$ and $|f(z)| \leq Ce^{\mu x}$ for $x \leq 0$, where $x = \Re z$ and C is a constant. It then follows from the Paley-Wiener theorem that there is a distribution μ supported on $[\mu, \lambda]$ such that (1) holds. More precise asymptotic estimates for f on the imaginary axis then show that μ is a finite measure, and further it can be decomposed as $d\mu = d\mu_s + \psi(t)dt$ where μ_s is a sum of point masses at the eigenvalues of B, and ψ is a piecewise continuous function un $[\mu, \lambda]$ whose only possible discontinuities are jumps at the eigenvalues of B.

From the asymptotics one obtains an explicit expression for μ_s which shows that it is a positive measure. Then the conjecture is equivalent to $\psi(t) \geq 0$ for all t.

A simple case is when A and B commute. Then they can be simultaneously diagonalised by a unitary transformation, which does not change $f_{A,B}$, so we can assume $A = \mathrm{diag}(a_1, \cdots, a_d)$ and $B = \mathrm{diag}(b_1, \cdots, b_d)$. Then $f(z) = \sum_j e^{a_j + zb_j}$ and $\mu = \sum e^{a_j} \delta_{b_j}$ so in this case ψ is identically zero.

It is useful to note that translation of A and B, and scaling of B, have simple effects on $\mu_{A,B}$. For if $A' = A + \alpha I$ and $B' = \gamma B + \beta I$, where α, β and γ are real constants, then $f_{A',B'}(z) = e^{\alpha + \beta z} f_{A,B}(\gamma z)$ and hence $\mu_{A',B'} = e^{\alpha} g(\mu_{A,B})$ where the mapping g of $\mathbb{R}$ is given by $g(t) = \beta + \gamma t$.

Another case in which μ can be found explicitly is when $d = 2$. This case is trivial if B has equal eigenvalues, and otherwise one can, by unitary transformation and translation and scaling of the form described in the previous paragraph, reduce to the case $A = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R}$. By calculating the eigenvalues of $A + zB$ we find $f(z) = 2e^{(a+z)/2} \cosh(\frac{1}{2}\sqrt{(z+a)^2 + 4b^2})$ and then from tables of Laplace transforms we find that $\mu_s = \delta_0 + e^a \delta_1$ and ψ can be expressed in terms of a Bessel function, or as a power series $\psi(t) = e^{at} \sum_{n=0}^{\infty} \frac{b^{2n+1}(t-t^2)^n}{n!(n+1)!}$, $t \in (0, 1)$. It is clear from this series that $\psi(t) > 0$ so the BMV conjecture is true for 2×2 matrices.

Much more information can be found in Moussa [12]. The above calculation for 2×2 matrices does not extend easily to larger values of d, and the conjecture remains open for 3×3 and larger matrices. However, numerical calculations of ψ by Grafendorfer [5] support the conjecture for the 3×3 case. Much of the progress on the conjecture uses a different but equivalent formulation, which we now motivate.

To prove the conjecture it suffices, using translation of A and B, to prove it when A and B are positive semi-definite (psd), so that μ lives on the positive real axis. Then, using a classical theorem of Bernstein on Laplace transforms, to show μ is a positive measure, it suffices to show that f and all its derivatives are non-negative on $[0,\infty)$. In fact, using translation again, it suffices to show they are non-negative at 0.

Now we can expand $f(z) = \sum_{k,l=0}^{\infty} \frac{1}{(k+l)!}\tau_{k,l}(A,B)z^l$ where $\tau_{k,l}(A,B)$ is the trace of the sum of all products of k A's and l B's. We want to show the coefficient of z^l is ≥ 0 for each l, and for this it suffices that $\tau_{k,l} \geq 0$ for all non-negative integers k and l. This is in effect the second version of the conjecture, which we state precisely and study in the next section,

2. Second Version

Let $T_{k,l} = T_{k,l}(A,B)$ be the sum of all products of k A's and l B's, so that for example $T_{2,1}(A,B) = AAB + ABA + BAA$, and then $\tau_{k,l}$ as defined at the end of the last section is given by $\tau_{k,l} = \mathrm{tr}(T_{k,l})$. The new version of the conjecture is:

BMV Conjecture: Version 2. If $d \in \mathbb{N}$ and A and B are psd $d \times d$ matrices, then

$$\tau_{k,l}(A,B) \geq 0 \tag{2}$$

for all $k, l \geq 0$.

An equivalent formuation which is often quoted is that, if $n \in N$ and we write $\phi_n(t) = \mathrm{tr}(A+tB)^n$, then ϕ_n is a polynomial whose coefficients are all nonnegative. In fact, $\phi_n(t) = \sum_{k=0}^{n} \tau_{n-k,k}(A,B)t^k$, so this indeed equivalent to Version 2.

The argument at the end of the last section shows that Version 2 implies Version 1. That it is in fact equivalent is proved in [10].

Version 2 appears to be more tractable, and partial results have been proved, to the extent that it is now known to be true for all k, l such that either $k \leq 4$, or $k \leq 6$ and $l \leq 8$ (or the same with k, l interchanged). We now outline the methods used to obtain these results.

'Easy' cases. $T_{k,l}$ is a sum of $\binom{k+l}{k}$ terms and there are two situations where one can show that each term has non-negative trace. The first is when $d = 2$; in this case we can find a unitary U so that the matrices U^*AU and U^*BU each have all entries ≥ 0. The same will then be true for all products of sequences of A's and B's, and hence such products have non-negative trace. This confirms that the BMV conjecture is true for 2×2 matrices, as already seen in Section 1.

The second situation is when either k or l is ≤ 2. Then any product of k A's and l b's has non-negative trace, as is easily seen by cyclic permutation. For example $\mathrm{tr}(ABA^3B) = \mathrm{tr}(aBA^3Ba)$ $= \mathrm{tr}(XX^*) \geq 0$, where $a = A^{1/2}$ and $X = aBAa$. So we conclude that when $\min(k,l) \leq 2$ we have $\tau_{k,l}(A,B) \geq 0$ for all psd A, B.

General observations. In studying $\tau_{k,l}(A,B)$ the are certain reductions that can be made. First, for any unitary U, $\tau_{k,l}(A,B) = \tau_{k,l}(U^*AU, U^*BU)$, which means one assume that A, say, is diagonal. Also, one can write $A = A' + \alpha I$ and $B = B' + \beta$ where $\alpha, \beta \geq 0$ and A', B' are psd and singular. Then we can write $\tau_{k,l}(A,B)$ as a linear combination, with positive coefficients, of terms $\tau_{r,s}(A', B')$ with $r \leq k$ and $s \leq l$. This allows one to reduce the conjecture to the case where A and B are singular.

We can also make use of the invariance of the trace under cyclic permutation, as we did in the second of the 'easy cases' above. We can group the individual products in $T_{k,l}(A,B)$ into equivalence classes under cyclic permutation. For example, $T_{3,3}(A,B)$ is a sum of 20 products which split into 4 equivalence classes, of sizes 6, 6, 6 and 2, and with representatives A^3B^3, ABA^2B^2, AB^2A^2B and

$(AB)^3$ respectively. So we have $\tau_{3,3}(A,B) = 6\text{tr}(A^3B^3)+6\text{tr}(ABA^2B^2)+6\text{tr}(AB^2A^2B)+2\text{tr}((AB)^3)$. Note that if k and l are relatively prime then all equivalence classes have size $k+l$.

We also note that if S is a product of A's and B's then S^* is the product in reverse order, and $\text{tr}(S^*)$ is the complex conjugate of $\text{tr}(S)$. Then in the above equivalence class decomposition, some classes will be 'self-conjugate', in the sense that S^* belongs to the class when S does, and the other classes can be grouped into pairs of conjugate classes. In the case of $\tau_{3,3}(A,B)$, the classes of (A^3B^3) and $(AB)^3$ are self-conjugate, whereas those of (ABA^2B^2) and (AB^2A^2B) form a conjuage pair. then we can write $\tau_{3,3}(A,B) = 6\text{tr}(A^3B^3) + 12\Re\text{tr}(ABA^2B^2) + 2\text{tr}((AB)^3)$.

First 'non-trivial' result. The first case of the conjecture to be solved, other than those covered by the above 'easy cases', was the proof in [6] that $\tau_{3,3}A,B \geq 0$ when A,B are 3×3 real psd matrices. The authors of this paper used the available reductions to treat the problem as one of positivity of a polynomial in a manageably small number of variables, and then proved this by fairly involved calculation.

Variational method. About the same time as [6], a variational method for attacking the the seond version of the BMV conjecture was developed in [3], following previous work on the first version [9, 12]. Although this method on its own has not resolved any cases of the conjecture, as we shall see it has been effective in combination with other approaches. A simpler version of this method was given in [8] and we now describe this in modified form.

By scaling, it is enough to treat the case when A and B both have trace 1. We then fix k,l,d and consider the problem of minimising $\tau_{k,l}(A,B)$ subject to A,B being psd $d\times d$ matrices of trace 1. As the set of such matrices is compact minimisers exist, and clearly to prove the conjecture for the chosen k,l,d it is enough to prove it for all minimisers.

Now suppose (A,B) is a minimiser, and let X be an arbitrary $d\times d$ matrix. For real t near 0 let $A(t) = (\text{tr}(A_0(t))^{-1}A_0(t)$ where $A_0(t) = (I+tX)^*A(I+tX)$. Then $A(t)$ is psd with trace 1, so $\phi(t) = \tau_{k,l}(A(t),B)$ has a minimum at 0, and $\phi'(t) = 0$. A calculation yields $\phi'(0) = (k+l)\text{tr}\{[X^*A+AX-\text{tr}(X^*A+AX)A]T_{k-1,l}(A,B)\}$. This is 0 for all X, from which one can readily deduce that $AT_{k-1,l} = \text{tr}(AT_{k-1,l})A = \frac{k}{k+l}\text{tr}(T_{k,l})A$. In the same way $BY_{k,l-1} = \frac{l}{k+l}\text{tr}(T_{k,l})B$. By scaling A and B we can deduce that, if the conjecture is false for a particular k,l,d then we can find non-zero psd A and B such that

$$AT_{k-1,l}(A,B) = -kA, \qquad BT_{k,l-1}(A,B) = -lB \tag{3}$$

One might hope to prove the conjecture by obtaining a contradiction from (3). That has not been achieved; the most useful result so far obtained from (3) is the following. We have, assuming (3), that $T_{k,l} = AT_{k-1,l}+BT_{k,l-1} = -kA-lB$ and then $\tau_{k+1,l} = \frac{k+l+1}{k+1}\text{tr}(AT_{k,l}) = \frac{k+l+1}{k+l}\text{tr}(A(-kA-lB)) < 0$, which means that the conjecture fails for $k+1,l,d$ and similarly for $k,l+1,d$. So failure for a particular k,l implies failure for all k',l' with $k'\geq k$ and $l'\geq l$. So to prove the conjecture, it is enough to prove it for a sequence of k_n,l_n with $k_n\to\infty$ and $l_n\to\infty$. This result may not seem too impressive, as one might expect the problem to get harder as k and l increase, but as we shall see it turns out to be surprisingly useful.

The latest developments. In 2007 Hägele [4] published a remarkably simple proof that $\tau_{4,3}(A,B)\geq 0$ for any positive semidefinite matrices A,B (of any size). It goes as follows: first note that $T_{4,3}$ is a sum of 35 products, which split into 5 equivalence class under cyclic permutation, each of size 7. Then we can express the trace by choosing a representative from each class:

$$\tau_{4,3} = 7\text{tr}(BA^2BA^2B+A^2B^3A^2+A^2B^2ABA+ABAB^2A^2+ABABABA) = 7\text{tr}(uBu*+vBv*)\geq 0$$

where $u = BA^2$ and $v = A^2B+ABA$.

Using the conclusion from the variational method, this also solves the $k = l = 3$ case. Hägele also showed that his method does not work directly for $k = l = 3$, thereby illustrating the usefulness of the variational approach.

It is natural to ask how far this approach can be pushed, and this has been explored in 3 more recent papers [8, 10, 2]. Hägele's method also works for the case $k = 5$, $l = 4$; in fact $\tau_{5,4} = 9\mathrm{tr}(uBu^* + vBv^* + wBw^*)$ where $u = BA^2B$, $v = B^2A^2 + BABA$ and $w = aB^2A + ABAB + A^2B^2$. As shown in [10] this extends in a natural way to show $\tau_{k,4} \geq 0$ whenever k is odd, and in [2] the argument is modified to show the same for k even. Applying the result of the variational analysis we the deduce that the conjecture holds for all k, l with $\min(k, l) \leq 4$.

To proceed further we formalise the Hägele approach. Let $\mathcal{A}$ denote the algebra over $\mathbb{R}$ of formal linear combinations of words in the symbols A, B. Let $\mathcal{P}$ denote the cone in $\mathcal{A}$ consisting of all linear combinations, with non-negative coefficients, of elements of the form uu^*, uAu^*, uBu^* and $AuBu^*$ for $u \in \mathcal{A}$. Let $\mathcal{C}$ be the linear span of all commutators $uv - vu$ for $u, v \in \mathcal{A}$. Finally let $\mathcal{Q} = \mathcal{P} + \mathcal{C} = \{u + v : u \in \mathcal{P}, v \in \mathcal{C}\}$.

Then if $T_{k,l} \in \mathcal{Q}$ it follows that $\tau_{k,l}(A, B) \geq 0$ for all positive semidefinite A, B. Hägele's argument is a proof that $T_{4,3} \in \mathcal{Q}$. It is then natural to ask: for which pairs (k, l) is $T_{k,l} \in \mathcal{Q}$? This is partly answered in the papers [8, 10, 2]. It is elementary that $T_{k,l} \in \mathcal{Q}$ if $\min(k, l) \leq 2$. As observed above, it is shown in [10, 2] that $T_{k,4} \in \mathcal{Q}$ for all k. It is also shown in [10] that $T_{8,3} \in \mathcal{Q}$, and, apart from the cases just mentioned, there is no other such pair with either k or l odd.

This leaves the case where k and l are both even. In [8] a procedure called semidefinite programming (see below) is used to show that $T_{8,6} \in \mathcal{Q}$. This is the only both-even case with $k, l > 4$ which has been definitively settled, though it is reported in [8] that numerical evidence indicates that $T_{6,6}$, $T_{10,6}$, $T_{8,8}$ and $T_{12,6}$ are not in $\mathcal{Q}$. The $T_{8,6}$ case together with the variational result proves the conjecture for all k, l such that $k \leq 8$ and $l \leq 6$ (or vice versa), in addition to the case $\min(k, l) \leq 4$ already mentioned. To prove the conjecture in general it suffices to find a sequence (k_n, l_n) with $k_n, l_n \to \infty$ and $T_{k_n,l_n} \in \mathcal{Q}$, and it is possible that such a sequence exists, with k_n and l_n even, although the results of [8] are not too encouraging in this regard.

Semidefinite programming. We indicate how the question $T_{k,l} \in \mathcal{Q}$ can be formulated in a manner suitable for the application of semidefinite programming. To do this we consider the case $T_{4,3}$ studied in [4]. To show $T_{4,3} \in \mathcal{Q}$ we try to find a real positive semidefinte matrix (c_{ij}) and words u_i in A, B so that $\sum c_{ij} u_i B u_j^*$ consists of exactly one representative of each equivalence class of terms in $T_{4,3}$; then $7 \sum c_{ij} u_i B u_j^* \in \mathcal{P}$ and differs from $T_{4,3}$ by a linear combination of commutators, so $T_{4,3} \in \mathcal{Q}$.

To get the right number of A'a and B's in the products we need two A'a and one B in each c_i, so we take $c_1 = AAB$, $c_2 = ABA$ and $c_3 = BAA$. We then find that to get the right representation of each of the 5 equivalence classes we need $c_{11} = c_{12} = c_{21} = 1$, $c_{22} + 2c_{23} = 1$ and $c_{33} + 2c_{13} = 1$, in other words the matrix $C = (c_{ij})$ is of the form $C = \begin{pmatrix} 1 & 1 & b \\ 1 & 1-2c & c \\ b & c & 1-2b \end{pmatrix}$ for some $b, c \in \mathbb{R}$. The problem then is to find values of b and c so that C is positive semidefinite. The simplest such choice is $b = c = 0$, which gives Hegele's construction, but it is not unique - in fact it is not hard to see that the set of suitable (b, c) is a region in the half-plane $c \leq 0$ bounded by a simple closed curve with equation $2b^2c + 6bc - b^2 - c^2 - 2c = 0$.

For general k, l the question of whether $T_{k,l} \in \mathcal{Q}$ reduces in the same way to the existence of a psd matrix whose coefficients satisfy a set of linear equations. This is the type of problem studied in semidefinite programming. In fact it treats the more general problem of find a psd matrix which minimises a given functional subject to a set of linear constraints. We are only interested in the existence of such a matrix satisfying the constraints, i.e. whether the feasible region is non-empty. In the case of $T_{4,3}$ the matrix is small enough that this question is quite easy to answer. A somewhat harder example is given by $T_{8,3}$ which is shown to be in $\mathcal{Q}$ in [10]. In this case we look for a 5×5 psd matrix C such that $\sum c_{ij} u_i B u_j^*$ consists of exactly one representative of each equivalence class, where now $u_1 = A^4 B$, $u_2 = A^3 BA$ etc. There are 15 equivalence classes giving 15 constraints, which taking account of the symmetry of C reduce to 10 equations $c_{11} = c_{12} = c_{13} = c_{14} = c_{22} = c_{23} = 1$, $c_{33} + 2c_{35} = 1$, $c_{44} + 2c_{34} = 1$, $c_{55} + 2c_{15} = 1$ and $c_{24} + c_{25} + c_{45} = 1$. The problem is simplified a little by the observation that for a psd matrix the conditions $c_{11} = c_{12} = c_{22} = 1$ force the first two rows to be identical, and we see then that what we need is a psd matrix of the form shown on the right (we omit the first row and column) for some $a, b, c \in \mathbb{R}$. One can show that there is a convex region with nonempty interior in (a, b, c) space for which this matrix is psd. A particular choice is $a = -1$, $b = -5$, $c = -3$. It is of interest to note that in this case, in contrast to the $T_{k,4}$ case, there are no possible choices with all $c_{ij} \geq 0$.

$$\begin{pmatrix} 1 & 1 & 1 & a \\ 1 & 1-2c & b & c \\ 1 & b & 1-2b & -a \\ a & c & -a & 1-2a \end{pmatrix}$$

The above examples involve small matrices for which it is practical to do calculations by hand. For $T_{8,6}$ much larger matrices are involved, and the numerical methods of semidefinite programming are required. Moreover, when k and l are even there are 3 different types of possible expansion rather than the one that occurs when either is odd. So instead of the single expansion $\sum c_{ij} u_i B u_j^*$ used above, for $T_{8,6}$ we consider $\sum c_{ij} u_i u_j^* + \sum d_{ij} A v_i A v_j^* + \sum e_{ij} B w_i B w_j^*$ where C, D, E are respectively 35×35, 20×20 and 15×15 psd matrices and $u_1, \cdots u_{35}$ is an enumeration of the products with 4 A's and 3 B's, $v_1, \cdots v_{20}$ is an enumeration of the products with 3 A's and 3 B's and $w_1, \cdots w_{15}$ is an enumeration of the products with 4 A's and 2 B's. There are 217 equivalence classes each giving a constraint which is a linear relation possibly involving coefficients of all three matrices; the symmetry reduces the number of constraints to 126. The solution found in [8] in fact has $D = 0$, so only two matrices are needed, one 35×35 and one 15×15. Information on the methodology of semidefinite programming can be found in [13].

Another approach. Although the possibility of proving the BMV conjecture by showing $T_{k,l} \in \mathcal{Q}$ for arbitrarily large pairs (k, l) is not ruled out, the indications from [8] of negative results for the pairs $(6, 6)$, $(8, 8)$, $(10, 6)$ and $(12, 6)$ are not too encouraging. We propose a modification of this approach, making more use of the variational result, which may give a better chance of success.

Recall that by the variational result, if the conjecture is false for a pair (k, l), then we can find psd matrices A and B such that $AT_{k-1,l} = T_{k-1,l}A = -kA$ and $BT_{k,l-1} = T_{k,l-1}B = -lB$. With this in mind, we let $\mathcal{I}_{kl}$ be the (2-sided) ideal in $\mathcal{A}$ generated by $AT_{k-1,l} + kA$, $T_{k-1,l}A + kA$, $BT_{k,l-1} + lB$ and $T_{k,l-1}B + lB$. We also define $\mathcal{P}^+$ to be the cone consisting of those $u = u(A, B) \in \mathcal{A}$ such that $\mathrm{tr}u(A, B) > 0$ whenever A and B are psd matrices with $AB \neq 0$, and then let $\mathcal{Q}^+ = \mathcal{P}^+ + \mathcal{C}$. Note that if $u \in \mathcal{Q}^+$ and $v \in \mathcal{Q}$ then $u + v \in \mathcal{Q}^+$, and also if $\min(r, s) \leq 2$ then any product u of r A's and s B's is in $\mathcal{Q}^+$.

Now if the conjecture is false for (k, l) and A and B are given by the variational method, then $u(A, B) = 0$ for any $u \in \mathcal{I}_{kl}$. On the other hand $AB \neq 0$ and so $\mathrm{tr}u(A, B) > 0$ for any $u \in \mathcal{Q}^+$. Hence $\mathcal{I}_{kl} \cap \mathcal{Q}^+ = \emptyset$.

So to prove the conjecture it suffices to prove that $\mathcal{I}_{kl} \cap \mathcal{Q}^+ \neq \emptyset$ for arbitrarily large pairs (k, l). The point here is that the requirement $\mathcal{I}_{kl} \cap \mathcal{Q}^+ \neq \emptyset$ is weaker than the previous requirement $T_{k,l} \in \mathcal{Q}$. In other words if $T_{k,l} \in \mathcal{Q}$ then $\mathcal{I}_{kl} \cap \mathcal{Q}^+ \neq \emptyset$ but the converse is false, in fact $\mathcal{I}_{33} \cap \mathcal{Q}^+ \neq \emptyset$ while $T_{3,3} \notin \mathcal{Q}$. This raises the possibility that there may be a better chance of proving $\mathcal{I}_{kl} \cap \mathcal{Q}^+ \neq \emptyset$ (for arbitrarily large pairs) than that $T_{kl} \in \mathcal{Q}$.

To justify the claims in the last paragraph, note first that if $T_{kl} \in \mathcal{Q}$ then $kA + lB \in \mathcal{P}^+$ so if we write $u = kA + lB$ then $u \in \mathcal{Q}^+$, and also $u = kA + lB + AT_{k-1,l} + BT_{k,l-1} \in \mathcal{I}_{kl}$. On the other hand, $\mathcal{I}_{33} \cap \mathcal{Q}^+ \neq \emptyset$, because $AT_{33} + 3A^2 + 3AB = A\{(T_{23} + 3A) + (T_{32} + 3B)\} \in \mathcal{I}_{33}$ and it is also in $\mathcal{Q}^+$ since $\mathrm{tr}(AT_{33}) = \frac{4}{7}\mathrm{tr}(T_{43}) \geq 0$ for any psd A and B, using $T_{43} \in \mathcal{Q}$.

To illustrate the possible use of the new approach, consider the case $k = l = 7$, which is the simplest case still open. If for example one could show that, for some choice of nonnegative $\alpha, \beta, \gamma, \delta$, not all zero, we have $u = \{(\alpha(A^2 + BA^2) + \beta ABA\}T_{6,7} + \{\gamma(B^2A + AB^2) + \delta BAB\}T_{7,6} \in \mathcal{Q}$, then it would follow that $7u + (2\alpha + \beta)ABA + (2\gamma + \delta)BAB \in (\mathcal{I}_{77} + \mathcal{C}) \cap \mathcal{Q}^+$, so that the conjecture holds for $k = l = 7$. It is possible that the freedom to choose $\alpha, \beta, \gamma, \delta$ may make this more likely than, say, showing that $T_{8,8} \in \mathcal{Q}$.

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