TRANSVERSALITY IN ALGEBRA AND TOPOLOGY I.

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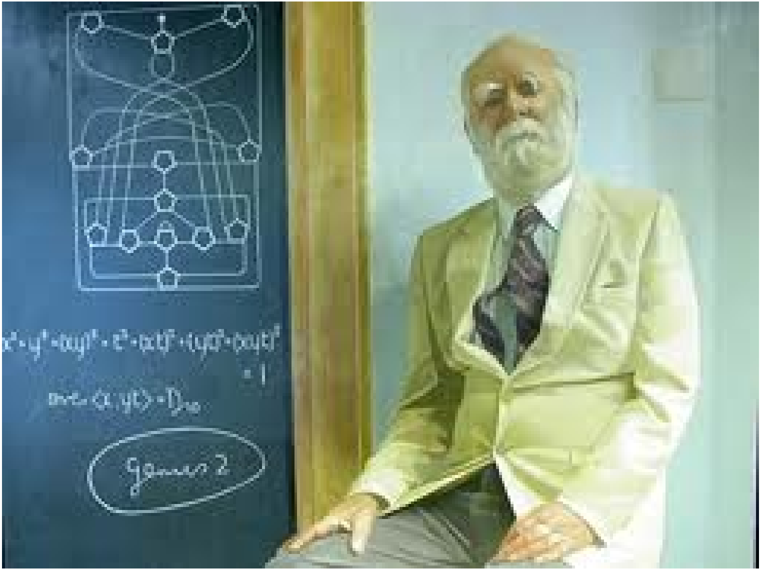
J.H.C. Whitehead, 1904-1960

Graham Higman, 1917-2008



The units of group-rings

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THE UNITS OF GROUP-RINGS

By GRAHAM HIGMAN.

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1. *Introduction.*

Let G be any group, and K any ring. The formal sums

$$r_1 e_1 + r_2 e_2 + \ldots + r_n e_n, \quad r_i \in K, \quad e_i \in G \quad (i = 1, 2, \ldots, n), \tag{1.1}$$

when addition and multiplication are defined in the obvious way, form a ring, the group-ring of G over K, which will be denoted by $R(G, K)$. Hence-

An invertible matrix (a_{ij}) over $\mathbb{Z}[G]$ represents 0 in the Whitehead group $Wh(G)$ if ...

Section 5 is concerned with a condition on G which can be expressed as follows: Consider the following transformations on a matrix $\|a_{ij}\|$, $i, j = 1, \ldots, n$, whose elements are in $R(G, C)$:

(*a*) Multiplying a row on the left, or a column on the right by $\pm e$; that is, replacing the row $(a_{i1}, \ldots, a_{in})$ by $(\pm ea_{i1}, \ldots, \pm ea_{in})$, or the column $(a_{1i}, \ldots, a_{ni})$ by $(\pm a_{1i}e, \ldots, \pm a_{ni}e)$, $e \in G$;

1.2) (*b*) Adding to one row (column) of $\|a_{ij}\|$ a left (right) multiple of another row (column) by an element in $R(G, C)$; that is, replacing the row $(a_{i1}, \ldots, a_{in})$ by $(a_{i1}+ra_{j1}, \ldots, a_{in}+ra_{jn})$, or the column $(a_{1i}, \ldots, a_{ni})$ by $(a_{1i}+a_{1j}r, \ldots, a_{ni}+a_{nj}r)$, $i \neq j$, $r \in K$;

(*c*) Bordering $\|a_{ij}\|$ with a row and a column of zeros meeting in a 1; that is, putting $a_{in+1} = a_{n+1i} = 0$, $i = 1, \ldots, n$, $a_{n+1n+1} = 1$; or the inverse of this.

Then the condition on G is that if $\|a_{ij}\|$ has a left inverse $\|a^*_{ij}\|$, so that

$$(1.3) \qquad \sum_{r=1}^{n} a^*_{ir} a_{rj} = \delta_{ij} \quad (i, j = 1, \ldots, n),$$

then $\|a_{ij}\|$ is transformable into the one-rowed matrix $\|1\|$ by a sequence of transformations (1.2).

$Wh(\mathbb{Z}) = 0$ I. Every invertible matrix (a_{ij}) over $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[x, x^{-1}]$ represents 0 in $Wh(\mathbb{Z})$

THEOREM 15. *The matrix condition of § 1 holds for the free cyclic group.*

Let $\|a_{ij}\|$ be a matrix whose elements are in $R(G, C)$, where G is the free cyclic group generated by x. Obviously we can transform $\|a_{ij}\|$ by means of transformations (1.2) so that it contains only positive powers of x. Moreover we can transform it so that it contains no power of x higher than the first. For let x^n, $n > 1$, be the highest power of x in $\|a_{ij}\|$, and

$Wh(\mathbb{Z}) = 0$ II. The Higman linearization trick: every element in $Wh(\mathbb{Z})$ is represented by a linear matrix $(a_{ij}) = (b_{ij} + c_{ij}x)$

let $a_{ij} = p_{ij}x^n + q_{ij}$, where q_{ij} contains no power higher than the $(n-1)$-th, and p_{ij} is an integer. Then we have

$$\|p_{ij}x^n+q_{ij}\| \to \left\|\begin{array}{c|c} p_{ij}x^n+q_{ij} & 0 \\ \hline 0 & \delta_{ij} \end{array}\right\| \to \left\|\begin{array}{c|c} p_{ij}x^n+q_{ij} & \delta_{ij}x \\ \hline 0 & \delta_{ij} \end{array}\right\|$$

$$\to \left\|\begin{array}{c|c} q_{ij} & \delta_{ij}x \\ \hline -p_{ij}x^{n-1} & \delta_{ij} \end{array}\right\|,$$

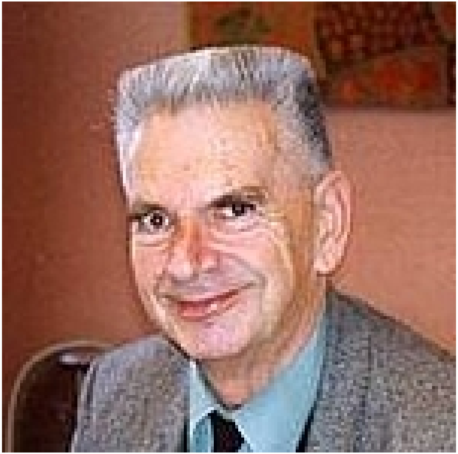
and the assertion follows by induction on n.

Suppose, therefore, that $a_{ij} = b_{ij} + c_{ij}x$, where b_{ij} and c_{ij} are integers. Since the ordinary elementary transformations on integer matrices† are a particular case of transformations (1.2), we may suppose either that $\|b_{ij}\|$ or that $\|c_{ij}\|$ has diagonal form. But if $\|a_{ij}\|$ has an inverse, the determinant $|a_{ij}|$ has the value $\pm x^p$, so that at least one of $|b_{ij}|$, $|c_{ij}|$ is zero. If $|b_{ij}|$ is zero suppose that $\|b_{ij}\|$ is reduced to diagonal form; otherwise suppose that $\|c_{ij}\|$ is reduced to diagonal form. Then some column of $\|a_{ij}\|$, which we may take to be the last, consists either entirely of integers or entirely of integer multiples of x. In the latter case multiply it by x^{-1}. By a further manipulation of rows, this column can be made to take the form

$$(0, 0, \ldots, 0, \lambda),$$

where, since $\|a_{ij}\|$ has an inverse, $\lambda = \pm 1$. Hence the order of the matrix can be reduced without destroying its linearity. By an induction on order the theorem follows.

René Thom, 1923-2002



Thom's 1954 paper on cobordism

Commentarii Mathematici Helvetici 28, 17-86 (1954)

Quelques propriétés globales des variétés différentiables

Par RENÉ THOM, Strasbourg

Introduction

Le présent article donne la démonstration des résultats que j'ai annoncés dans quatre Notes aux Comptes-Rendus [28][1]). Il est divisé en quatre chapitres. Le premier chapitre élabore une technique d'approximation des applications différentiables ; les théorèmes démontrés sont en quelque sorte une formulation différentiable du théorème d'approximation simpliciale de la Topologie ; grâce à eux, toute la théorie pourra être établie sans faire appel au théorème de triangulation des variétés différentiables. Le chapitre II est consacré au problème de la

The statement of the Thom transversality theorem

Théorème I. 5. *Soit f une application de classe C^n de la variété V^n dans la variété M^p, et soit N^{p-q} une sous-variété paracompacte de M^p, et T un voisinage tubulaire normal de N^{p-q} dans M^p. Il est possible de trouver un homéomorphisme A de T sur lui-même, arbitrairement voisin de l'identité dans* $\mathbf{H}$, *tel que, si $f' = A \circ f$:*

i) *L'image réciproque $f'^{-1}(N^{p-q})$ de N^{p-q} soit une sous-variété de V^n, de dimension $n-q$: W^{n-q} différentiablement plongée de classe C^n.*

ii) *L'espace fibré des vecteurs normaux à W^{n-q} dans V^n soit canoniquement isomorphe à l'espace induit de la structure fibrée des vecteurs normaux à N^{p-q} dans M^p.*