1 What is an infinite cyclic cover?

1.1 Covers and their properties

Definition 1.1. Given a topological space $X$, a continuous map $p: Y \to X$ is a covering map if each $x \in X$ has an open neighbourhood $U$ such that $p^{-1}(U)$ is a nonempty disjoint union of open sets, and such that the restriction of $p$ to each of these open sets is a homeomorphism.

Terminology. $Y$ is called a covering space of $X$ if there exists a covering map $p: Y \to X$.

For $x \in X$, $p^{-1}\{x\}$ is called the fibre over $x$. It has the discrete topology.

Each such neighbourhood $U$ is said to be evenly covered.

The most useful property of covering maps is that they allow paths and homotopies to be lifted uniquely into the covering space, where hopefully we can use nice features of the space (such as simply-connectedness) to do useful things. The next few definitions and theorems will make this rigorous.

Definition 1.2. Let $p: Y \to X$ be a covering map and $f: A \to X$ be any map. A lift of $f$ is a map $f': A \to Y$ such that $pf' = f$. In other words, the following diagram commutes:

\[\begin{array}{ccc}
A & \to & X \\
\downarrow{f} & & \downarrow{p} \\
Y & \to & \\
\end{array}\]

Lemma 1.3 (Uniqueness of lifting). Let $A$ be connected, $p: Y \to X$ be a covering map, $f: A \to X$ continuous and $f'_1, f'_2: A \to Y$ be two lifts of $f$. Then $\{a \in A | f'_1(a) = f'_2(a)\} = \emptyset$ or $A$.

Proof. Set $A' := \{a \in A | f'_1(a) = f'_2(a)\}$ and $D := A \setminus A'$. We will prove that $A'$ and $D$ are both open, hence by connectedness of $A$ one of them must be empty.

So fix $a \in A$ and let $U$ be an evenly covered neighbourhood of $f(a)$. Let $S_1, S_2$ be the open sets in $p^{-1}(U)$ containing $f'_1(a), f'_2(a)$ respectively. Then $V := f'^{-1}_1(S_1) \cap f'^{-1}_2(S_2)$ is an open neighbourhood of $a$. Let $z \in V$. 
If \( a \in A' \) then \( f^1_1(y) = f^2_1(y) \), so \( S_1 = S_2 \). So \( f^1_1(z), f^2_1(z) \in S_1 \), and \( pf^1_1(z) = f(z) = pf^2_1(z) \). But \( p|_{S_1} \) injects, so \( f^1_1(z) \neq f^2_1(z) \) and \( z \in A' \). Thus \( V \subset A' \), so \( A' \) is open.

If \( a \notin A' \) then \( a \in D \), in which case \( S_1 \neq S_2 \), and \( S_1 \cap S_2 = \emptyset \). So \( f^1_1(z) \neq f^2_1(z) \) and so \( z \in D \).

Thus \( V \subset D \), so \( D \) is open.  

\[ \text{Theorem 1.4 (Path-lifting property).} \]

If \( p: Y \to X \) is a covering map, and \( x_0 \in X \), \( y_0 \in Y \) are points such that \( p(y_0) = x_0 \), then every path \( \alpha: I \to X \) with \( \alpha(0) = x_0 \) has a unique lift to a path \( \tilde{\alpha}: I \to Y \) with \( \tilde{\alpha}(0) = y_0 \).

\[ \text{Proof.} \]

Uniqueness is clear from Lemma 1.3.

\( X \) has a cover by evenly covered open sets, so by the Lebesgue Covering Lemma we can find a partition \( 0 = t_0 < \cdots < t_N = 1 \) with each \( \alpha|_{[t_i, t_{i+1}]} \) lying in an evenly covered open set \( U_i \). Let \( y_i \) be in the fibre over \( \alpha(t_i) \) (i.e. \( p(y_i) = \alpha(t_i) \)) and let \( S \) be the open set in \( p^{-1}(U_i) \) with \( y_i \in S \).

Now \( p|_S: S \to U_i \) is a homeomorphism with inverse \( \psi: U_i \to S \), so define \( \tilde{\alpha}|_{[t_i, t_{i+1}]} \) by \( \tilde{\alpha}|_{[t_i, t_{i+1}]} = \psi\alpha|_{[t_i, t_{i+1}]} \).

We now proceed by induction.

Induction hypothesis: There is a lift \( \tilde{\alpha} \) defined on \( [0, t_1] \) with \( \tilde{\alpha}(0) = y_0 \).

Case \( i = 1 \): Take \( \tilde{\alpha}_1 = \tilde{\alpha}|_{[0, t_1]} \) as in the construction above.

Assume the hypothesis is true for some \( i \), and define \( \tilde{\alpha}_{i+1} \) on \( [0, t_{i+1}] \) by

\[ \tilde{\alpha}_{i+1}(s) = \begin{cases} \tilde{\alpha}_i(s) & s \in [0, t_i] \\ \overline{\alpha_\gamma(t_i)} & s \in [t_i, t_{i+1}] \end{cases} \]

This is well-defined and continuous, and \( p\tilde{\alpha}_{i+1} = \alpha \). Therefore by induction we may take \( \tilde{\alpha} = \tilde{\alpha}_N \).

\[ \text{Theorem 1.5 (Homotopy lifting property).} \]

Let \( F: I \times I \to X \) be a homotopy of paths, so \( F(t, 0) = \alpha(t) \) and \( F(t, 1) = \beta(t) \), and \( p: Y \to X \) be a covering map as before. Then there is a unique lift \( \tilde{F}: I \times I \to Y \) with \( \tilde{F}(t, 0) = \tilde{\alpha}(t) \) and \( \tilde{F}(t, 1) = \tilde{\beta}(t) \).

\[ \text{Proof.} \]

Similar to Theorem 1.4. See any standard textbook on Algebraic Topology for details.

We now put the Path-lifting property to some use, proving a useful theorem about fibres.

\[ \text{Proposition 1.6.} \]

Let \( p: Y \to X \) be a covering map, and let \( x_1, x_2 \) be in the same path-component of \( X \). Then \( p^{-1}\{x_1\} \) is homeomorphic to \( p^{-1}\{x_2\} \).

\[ \text{Proof.} \]

Let \( \alpha: I \to X \) be a path from \( x_1 \) to \( x_2 \). Then for each element of the fibre \( p^{-1}\{x_1\} \), \( \alpha \) lifts to a unique path \( \tilde{\alpha} \) ending at a unique point in \( p^{-1}\{x_2\} \). This gives us a continuous map \( p^{-1}\{x_1\} \to p^{-1}\{x_2\} \). Doing the same thing with the inverse path \( \alpha^{-1}(t) \) gives us a continuous map \( p^{-1}\{x_2\} \to p^{-1}\{x_1\} \). Since \( \tilde{\alpha}(0) = \tilde{\alpha}^{-1}(1) \) if and only if \( \tilde{\alpha}(1) = \tilde{\alpha}^{-1}(0) \), we see that these maps are also bijections and we are done.

Thus if \( X \) is path-connected it makes sense to talk about the fibre of the covering space.
Example 1.7. If \( X = S^1 \), the unit circle, we can define a covering map \( p: \mathbb{R} \to S^1 \) by \( p(t) = e^{2\pi i t} \). The fibre of this cover is easily seen to be \( \mathbb{Z} \).

This example is one of the most important in Algebraic Topology, and we would like to investigate spaces with similar kinds of covering maps. This motivates the following definition.

Definition 1.8. An infinite cyclic cover of a path-connected space \( X \) is a covering space with fibre \( \mathbb{Z} \).

1.2 Pullback covers

Here is one way of constructing new covers from old.

Definition 1.9. Let \( p: Y \to X \) be a covering space, and let \( f: A \to X \) be a continuous map. The pullback cover of \( Y \) by \( f \) is defined to be the space

\[
f^*Y = \left\{(a, y) \in A \times Y \mid f(a) = p(y)\right\}.
\]

So we have a commutative diagram

\[
\begin{array}{ccc}
  f^*Y & \xrightarrow{q_1} & Y \\
  q_2 \downarrow & & \downarrow p \\
  A & \xrightarrow{f} & X
\end{array}
\]

where \( q_1 \) and \( q_2 \) are just the canonical projection maps.

Proposition 1.10. The fibre of a pullback cover is homeomorphic to the fibre of the original cover.

Proof. Let everything be as in the commutative diagram above. Take a fibre in \( f^*Y \), which has the form \( q_2^{-1}(a) \) for some \( a \in A \). Now

\[
q_2^{-1}(a) = \left\{(a, y) \in \{a\} \times Y \mid f(a) = p(y)\right\}
\]

Projection onto the second factor gives us a bijection to the set

\[
\left\{y \in Y \mid y \in p^{-1}\{f(a)\}\right\}
\]

which is simply the fibre over \( f(a) \). Continuity is not an issue because fibres have the discrete topology.

One important class of infinite cyclic covers are those which are obtained as pullback covers of \( \mathbb{R} \) by maps to \( S^1 \), i.e. spaces \( f^*\mathbb{R} \) where \( f: X \to S^1 \):

\[
\begin{array}{ccc}
  f^*\mathbb{R} & \xrightarrow{\mathbb{R}} & \mathbb{R} \\
  \downarrow & & \downarrow e^{2\pi i t} \\
  X & \xrightarrow{f} & S^1
\end{array}
\]
By Proposition 1.10 these covers have the same fibre as \( \mathbb{R} \) over \( S^1 \), which is \( \mathbb{Z} \), so they are certainly infinite cyclic covers. In fact, as we shall see later, every infinite cyclic cover of a space is isomorphic to a pullback cover, so actually these are all the covers we will ever need to know about.

1.3 Pullback Infinite cyclic covers of \( S^1 \)

A map from \( S^1 \) to \( S^1 \) is characterised by how many times it wraps around the circle and in what direction. Therefore every map \( S^1 \to S^1 \) is homotopic to a map \( f_d: z \mapsto z^d \) for some \( d \in \mathbb{Z} \).

Let \( \mathbb{R} \) be the universal cover of \( S^1 \), with covering map \( p: t \mapsto e^{2\pi it} \). Consider the pullback infinite cyclic cover of \( S^1 \) by \( f_d \):

\[
\begin{array}{c}
\mathbb{R} \\
\downarrow \\
S^1 \end{array} \quad \begin{array}{c}
f_d^*\mathbb{R} \\
\downarrow \\
S^1 \end{array} \quad \begin{array}{c}
p \\
\downarrow \\
S^1 \\
\end{array}
\]

So \( f_d^*\mathbb{R} = \{(z, t) \in S^1 \times \mathbb{R} | z^d = e^{2\pi it}\} \).

Let us first consider the case \( d = 2 \). We get \( f_2^*\mathbb{R} = \{(z, t) \in S^1 \times \mathbb{R} | z = \pm e^{i\pi t}\} \), which is two spirals as shown in the following diagram:

So for \( t \in [0, 1] \), the spirals together cover all of \( S^1 \), even though individually each spiral only covers half of the circle.

Now that we have seen the special case where \( d = 2 \), the general case is very easy to see. The degree \( d \) infinite cyclic cover of \( S^1 \) consists of \( d \) spirals, one for each of the \( d \) roots of \( e^{2\pi it} \). The spirals are equally spaced around the circle and never intersect as they are each spiralling upwards.
at the same rate. In time \( t = 1 \) the spirals will collectively cover \( S^1 \) although each individual spiral takes time \( d \) to cover the same area.

Thus \( f_d^* \mathbb{R} \) is homeomorphic to \( \mathbb{R} \sqcup \cdots \sqcup \mathbb{R}_d \) (\( d \) disjoint copies of \( \mathbb{R} \)) via the map \( t_k \mapsto (t_k, e^{2\pi i (t + k)}) \), where \( t_k \in \mathbb{R}_k \). This is clearly injective and surjective, and also clearly continuous as the inverse image of an open set will be a collection of open sets in the various \( \mathbb{R}_i \), which is open.

We can, in fact, find the degree of any infinite cyclic cover of \( S^1 \). This is defined as follows:

**Definition 1.11.** Let \( \overline{S^1} \) be an infinite cyclic cover of \( S^1 \) with covering map \( p \). Let \( \phi \colon [0, 1] \to S^1 \) be a path travelling once around \( S^1 \) starting and ending at 1. Then we can find a map (by the path-lifting property 1.4) \( \overline{p} \colon [0, 1] \to \overline{S^1} \) which makes the following diagram commute:

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{\phi} & S^1 \\
\downarrow{\overline{p}} & & \\
\overline{S^1} & \xrightarrow{p} & S^1
\end{array}
\]

Now \( \overline{p}(0), \overline{p}(1) \in p^{-1}(1) \cong \mathbb{Z} \), and we define the degree of \( \overline{S^1} \) to be \( \overline{p}(1) - \overline{p}(0) \in \mathbb{Z} \).

**Remark 1.12.** Notice that \( f_d^* \mathbb{R} \) has degree \( d \), but that a degree \( d \) map may not be isomorphic to \( f_d^* \mathbb{R} \), e.g. \( \mathbb{R} \) with the covering map \( e^{2\pi i t} \) has degree \( d \) but cannot be homeomorphic to \( f_d^* \mathbb{R} \) because they have different numbers of path-components.

## 2 Isomorphism Classes of Covers

**Definition 2.1.** Suppose that \( Y_1 \) and \( Y_2 \) are two infinite cyclic covers of the same space \( X \), with covering maps \( p_1 \) and \( p_2 \) respectively. Then we say \( Y_1 \cong Y_2 \) if there is a homeomorphism \( h \colon Y_1 \to Y_2 \) such that \( p_2 h = p_1 \).

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{h} & Y_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
X & \xrightarrow{p_2} & X
\end{array}
\]

**Terminology.** We write \( \text{Iso}(X) \) to mean the set of isomorphism classes of infinite cyclic covers of \( X \).

**Theorem 2.2.** Suppose \( X \) is path-connected. Then we can find bijections

\[
\text{Iso}(X) \cong [X, S^1] \cong H^1(X, \mathbb{Z})
\]

where \([X, S^1]\) denotes the set of homotopy classes of maps from \( X \) to \( S^1 \).

The proof of this shall be given in stages. First we construct a map from \([X, S^1]\) to \( \text{Iso}(X) \).
Proposition 2.3. Define $\Phi: [X, S^1] \to \text{Iso}(X)$ by $\Phi([f]) = f^*\mathbb{R}$, the pullback infinite cyclic cover of $\mathbb{R}$ by $f: X \to S^1$. This is a well defined map.

**Proof.** Suppose $f \simeq g$. We want to show that $f^*\mathbb{R} \cong g^*\mathbb{R}$.

First of all we have a homotopy $F: X \times I \to S^1$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Consider the pullback cover $F^*\mathbb{R} := \{(x, t, r) \in X \times I \times \mathbb{R} \mid F(x, t) = p(r)\}$:

$$
\begin{array}{ccc}
F^*\mathbb{R} & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow \\
X \times I & \overset{F}{\longrightarrow} & S^1
\end{array}
$$

Here we can identify $f^*\mathbb{R}$ with the subset of $F^*\mathbb{R}$ where $t = 0$. Likewise $g^*\mathbb{R}$ is identified with $t = 1$.

Now, for each $x \in X$ we have a path $F_x: I \to S^1$ from $f(x)$ to $g(x)$, where $F_x(t) = F(x, t)$. By the path-lifting property (Theorem 1.4) we can lift this uniquely to a path $\tilde{F}_{x,r}: I \to \mathbb{R}$ starting at $r \in \mathbb{R}$, such that

$$p\tilde{F}_{x,r}(t) = F_x(t) = F(x, t).$$

Notice that $(x, \tilde{F}_{x,r}(1)) \in g^*\mathbb{R}$.

There is now an obvious path from $f^*\mathbb{R}$ to $g^*\mathbb{R}$. Define $\alpha_{x,r}: I \to F^*\mathbb{R}$ by

$$\alpha_{x,r}(t) = ((x, t), \tilde{F}_{x,r}(t)).$$

Then $h: f^*\mathbb{R} \to g^*\mathbb{R}$ is given by $(x, r) \equiv \alpha_{x,r}(0) \to \alpha_{x,r}(1) \equiv (x, \tilde{F}_{x,r}(1))$. This is continuous because $F$ is continuous, and has continuous inverse $h^{-1}$ constructed similarly but using the homotopy $F(x, 1-t)$. So $h$ is a homeomorphism and it clearly commutes with the covering maps (which are just projection onto the first factor).

Before we can find the inverse of this map, we need a few more pieces of terminology.

**Definition 2.4.** A graph in a simplicial complex $X$ is a collection of vertices (0-cells) of $X$ and edges (1-cells) between these vertices (not necessarily all of them). A tree is a contractible graph (i.e. a connected graph with no loops). A tree is called maximal if it contains all the vertices of $X$.

**Lemma 2.5.** Every path-connected graph contains a maximal tree, and any two vertices are joined by a unique minimal path in this tree.

**Proof.** Let $X$ be a path-connected graph and choose a single vertex $X_0$. We proceed by induction. Set $Y_0 = X_0$, then assuming we have a tree $Y_i$ we construct $Y_{i+1}$ by adjoining every vertex that can be reached by attaching the closure of one edge to $Y_i$. Then $Y_i \subseteq Y_{i+1}$ and it is clear that $Y_{i+1}$ deformation retracts to $Y_i$. Let $Y = \bigcup Y_i$. Since $X$ is path-connected, every vertex must lie in $Y$.

We can deformation retract $Y$ onto $X_0$ by performing a retraction of $Y_{i+1}$ to $Y_i$ during the time interval $[1/2^{i+1}, 1/2^i]$. Therefore $Y$ is a maximal tree.
There is a unique minimal path from each vertex to $X_0$: suppose we have a vertex $Z \in Y_i - Y_{i-1}$, then in time-step $k$ move along an edge from a vertex in $Y_{i+1-k}$ to a vertex in $Y_{i-k}$. By construction this is only possible in one way, and $i$ is the minimal time it can be achieved. To join any two points, join each individually to $X_0$ and see where the paths overlap; delete these overlapped paths and the resulting path will be a minimal way to get between the vertices.

**Proposition 2.6.** There is a map $\text{Iso}(X) \to [X, S^1]$ which is the inverse of $\Phi$.

**Proof.** We assume for simplicity that $X$ is a simplicial complex. Let $\overline{X}$ be an infinite cyclic cover of $X$, and choose a maximal tree $T \subset X$. We are going to construct a map $f: X \to S^1$. First of all, send every vertex and every edge in $T$ to the point $1 \in S^1$ (any point will do, as $S^1$ is path-connected). If an edge $\sigma$ is not in $T$, then there is a unique minimal simplicial path $\sigma_T \subset T$ that connects its start and end points. Then $\sigma \cup \sigma_T$ is a loop in $X$, and we define $n_\sigma$ to be the degree of the pullback of $\overline{X} \to X$ to this loop.

Then let $f(\sigma)$ be the edge in $S^1$ starting at 1 and wrapping around $S^1$ $n_\sigma$ times.

We now need to extend this map consistently to higher simplices. So suppose $\tau$ is a 2-simplex with boundary $\partial \tau$, a loop in $X$. Now this boundary is homotopic to a constant map (nullhomotopic) in $X$ via the 2-simplex $\tau$ itself, and therefore $f(\partial \tau)$ is homotopic to a constant in $S^1$. So if we map $\tau$ to $1 \in S^1$, this is consistent with our previous definition of $f$ on the 1-simplices. Similarly, we may send all higher simplices to $1 \in S^1$.

Furthermore, if we had chosen another infinite cyclic cover $\overline{X} \cong \overline{X}'$, then we would have constructed the same map (up to homotopy) because the degrees of pullbacks of loops would be the same for either cover.

It remains to show that if we construct the pullback cover $f^*\mathbb{R}$ of the universal infinite cyclic cover of $S^1$, this is isomorphic to $\overline{X}$.