Local holomorphic Euler characteristic and instanton decay

E. Gasparim T. Köppe P. Majumdar

Abstract

We study the local holomorphic Euler characteristic $\chi(x,\mathscr{F})$ of sheaves near a surface singularity obtained from contracting a line ℓ inside a smooth surface Z. We prove non-existence of sheaves with certain prescribed numerical invariants. Non-existence of instantons on Z with certain charges follows, and we conclude that ℓ^2 poses an obstruction to instanton decay. A *Macaulay 2* algorithm to compute χ is made available at `http://www.maths.ed.ac.uk/~s0571100/Instanton/`.

1 Introduction

Let $\sigma\colon (\widetilde{X},D) \to (X,x)$ be a resolution of an isolated quotient singularity. Let $\widetilde{\mathscr{F}}$ be a reflexive sheaf on $\widetilde{X}$, set $\mathscr{F} := (\sigma_*\widetilde{\mathscr{F}})^{\vee\vee}$; notice that there is an embedding $\sigma_*\widetilde{\mathscr{F}} \hookrightarrow \mathscr{F}$. Then the *local holomorphic Euler characteristic* of $\widetilde{\mathscr{F}}$ at x is defined by

$$\chi\big(x,\widetilde{\mathscr{F}}\big) := \chi\big((\widetilde{X},D),\widetilde{\mathscr{F}}\big) := h^0\big(X;\ \mathscr{F}/\sigma_*\widetilde{\mathscr{F}}\big) + \sum_{i=1}^{n-1}(-1)^{i-1}h^0\big(X;R^i\sigma_*\widetilde{\mathscr{F}}\big)\,. \tag{1.1}$$

For the case when X is an orbifold, Blache [Bl] shows that:

$$\chi\big(\widetilde{X},\widetilde{\mathscr{F}}\big) = \chi\big(X,\mathscr{F}\big) + \sum_{x\in\operatorname{Sing}X}\chi\big(x,\widetilde{\mathscr{F}}\big)\,. \tag{1.2}$$

In this paper we consider rational surface singularities obtained by contracting a line $\ell \cong \mathbb{P}^1$ with $\ell^2 < -1$ inside a smooth surface. To calculate χ locally, it is enough to study sheaves on a small neighbourhood of the singular point, or on a small neighbourhood of the exceptional set of a resolution. We therefore consider the spaces $Z_k := \operatorname{Tot}\big(\mathscr{O}_{\mathbb{P}^1}(-k)\big)$.

We denote by X_k the space obtained from Z_k by contracting the zero-section ℓ to a point, and we let $\pi\colon Z_k \to X_k$ be the contraction map. Since we are interested in applications to instantons, we will consider sheaves E over Z_k with $c_1(E) = 0$. Then $E|_\ell$ splits by Grothendieck's lemma, and there exists an integer $j \geq 0$ called the *splitting type* of E such that $E|_\ell \cong \mathscr{O}(j) \oplus \mathscr{O}(-j)$. Set $Z_k^o := Z_k - \ell$. We make two simple observations about reflexive sheaves on Z_k.

Proposition 6.7. Let E be a rank-2 reflexive sheaf on Z_k with splitting type $\geq k$, then $\chi(x,E) \geq k-1$.

Proposition 4.1. Let E_1 and E_2 be $\mathfrak{sl}(2,\mathbb{C})$-bundles over Z_k with splitting types j_1 and j_2, respectively. There exists an isomorphism $E_1|_{Z_k^o} \cong E_2|_{Z_k^o}$ if and only if $j_1 \equiv j_2 \mod k$. In particular, E_1 can decay totally over Z_k if and only if $j_1 \equiv 0 \mod k$.

This paper consists of applications of the local holomorphic Euler characteristic to problems of existence and decay of instantons. We also discuss the Kobayashi–Hitchin correspondence over Z_k. We obtain, via discussion of the physical consequences and an *ad hoc* definition of stability (Definition 5.2), the following conclusions:

Proposition 5.4. There is a one-to-one correspondence between framed $SU(2)$-instantons on Z_k with local charge n and framed-stable $\mathfrak{sl}(2,\mathbb{C})$-bundles on Z_k with $\chi^{\mathrm{loc}} = n$.

Corollary 5.5. An $\mathfrak{sl}(2,\mathbb{C})$-bundle over Z_k represents an instanton if and only if its splitting type is a multiple of k.

Theorem 6.8. The minimal local charge of a nontrivial $SU(2)$-instanton on Z_k is $\chi_k^{\min} = k-1$. The local moduli space of (unframed) instantons on Z_k having fixed local charge $\chi_k^{\min}$ has dimension $k-2$.

Acknowledgements: We are honoured to contribute to this volume commemorating Fedya's 60th birthday and are grateful to Ludmil Katzarkov for giving us this opportunity.

The first author thanks the hospitality and generous support of the Mathematics Institute of the Universität Münster, where part of this work was carried out. We thank M. Stillman for assistance with *Macaulay 2*.

2 Elementary background on instantons

Given a principal $SU(2)$-bundle $P \to X$ over a Riemannian 4-manifold X with $c_2(P) = n > 0$, an *$SU(2)$-instanton of charge n on X* is a connection A on P minimising the the Yang–Mills functional

$$S_{\mathrm{YM}}(A) := \int_X \|F_A\|^2 \ ,$$

where F_A is the curvature of the connection A. The Yang–Mills equations are the Euler–Lagrange equations corresponding to the functional S_{YM}.

Being non-linear and of second order, the Yang–Mills equations are quite difficult to study. Luckily a linearisation can be obtained easily as follows: The Yang–Mills equation of motion is $D(A) \wedge F(A) = 0$. But since the Jacobi identity $D(A) \wedge *F(A) = 0$ always holds, any A satisfying $*F_A = \pm F_A$ solves the Yang–Mills equations of motion. A connection A whose curvature satisfies

$$*F_A = -F_A \ , \tag{ASD}$$

is called *anti-self-dual.* Hence, anti-self-dual connections minimise the Yang–Mills functional. For this reason the ASD equations may be seen as a "linear version" of the Yang–Mills equations. Subsequently, from the mathematical point of view, instantons have become synonymous to anti-self-dual connections.

Over a compact Kähler surface X, the Kobayashi–Hitchin correspondence (see [LT]) provides an interpretation of irreducible $SU(2)$-instantons of charge n as stable holomorphic $SL(2)$-bundles over X with second Chern class $c_2 = n$:

$$\left\{ \begin{array}{c} \text{irreducible } SU(2)\text{-instantons} \\ \text{of charge } n \end{array} \right\} \overset{\text{K.–H.}}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{stable } SL(2)\text{-bundles} \\ \text{with } c_2 = n \end{array} \right\} ,$$
$$\nabla = \bar{\partial} + \partial \longleftrightarrow \bar{\partial} \ . \tag{2.1}$$

In particular, when X is a ruled surface, the informal interpretation of instantons as weighted, point-like configurations of concentrated charge has a precise interpretation in terms of jumping lines: An instanton bundle on a ruled surface has a generic splitting $\mathcal{O}(a)\oplus\mathcal{O}(-a)$, which is the same for all but finitely many lines of the ruling, called the jumping lines. In this interpretation, the weight of the points in the configuration corresponds to the multiplicity of the jumps, and the topological charge (= the second Chern class) is given as the sum of all multiplicities.

Multiplicities of jumps on lines having $\ell^2 = 0$, i.e. on trivial families $\mathbb{C}P^1 \times \mathbb{C}$, are quite well understood (cf. [BHMM], [HM]). Here we study the behaviour of instantons around lines with negative self-intersection $\ell^2 = -k$. The case $k = 1$ corresponds to the blow-up of a surface at a point and was studied in several papers, e.g. [GO], [Ga1], [Kn]. Near a $-k$-line, with $-k < 0$, an instanton has two independent local numerical invariants: the *height* and the *width* (see Definition 2.3), whose sum gives the *multiplicity* or *local charge*. The label “local charge” comes from the translation into algebraic geometry via the Kobayashi–Hitchin correspondence:

In the first case, let $\widetilde{X}$ be a compact complex surface containing a -1-line, and let $\pi\colon \widetilde{X} \to X$ be the blow-down of ℓ to a point $x_0 \in X$. If E is a bundle on $\widetilde{X}$, then the local second Chern class of E near ℓ is by definition

$$c_2^{\mathrm{loc}}\big(\ell, E\big) := c_2\big(E\big) - c_2\big((\pi_* E)^{\vee\vee}\big)\;.$$

Thus, a local version of the Kobayashi–Hitchin correspondence justifies the terminology

$$\left\{ \begin{array}{c} \text{local charge of an} \\ \text{instanton} \end{array} \right\} \overset{\text{K.–H.}}{\Longleftrightarrow} \left\{ \quad \text{local } c_2 \text{ of a bundle} \quad \right\}.$$

Still near the -1-line, an application of Hirzebruch–Riemann–Roch gives

$$c_2^{\mathrm{loc}}\big(\ell, E\big) = h^0\big(X; (\pi_* E)^{\vee\vee}/\pi_* E\big) + h^0\big(X; R^1\pi_* E\big)\;, \tag{2.2}$$

which by (1.1) is the local holomorphic Euler characteristic of E near ℓ.

However, the analogue of (2.2) for a bundle near a $-k$-line, where $k \geq 2$, is a more complicated issue, simply because contracting such a line produces a singularity; and there exist at least three nonequivalent definitions of Chern classes for singular varieties. To avoid carrying this problem over to instantons, it is more convenient to simply consider the local holomorphic Euler characteristic, and set:

Definition 2.3. Let E be an instanton bundle over a smooth surface Z containing a $-k$ line ℓ and let $\pi\colon Z \to X$ be map that contracts of ℓ to a point. The *local charge of E around ℓ* is:

$$\chi^{\mathrm{loc}}\big(E\big) = \chi\big(\ell, E\big) := h^0\big(X; (\pi_* E)^{\vee\vee}/\pi_* E\big) + h^0\big(X; R^1\pi_* E\big)\;.$$

The right hand side contains two independent holomorphic invariants:
$\mathbf{w}(E) := h^0\big(X; (\pi_* E)^{\vee\vee}/\pi_* E\big)$ is called the *width* of the instanton and measures how far the direct image is from being a bundle;
$\mathbf{h}(E) := h^0\big(X; R^1\pi_* E\big)$ is called the *height* of the instanton and measures how far the bundle is from being a split extension.

From equation (1.2), $\chi\big(E\big) = \chi^{\mathrm{loc}}\big(E\big) + \chi\big((\pi_* E)^{\vee\vee}\big)$, so we can say that the local charge measures the loss of total charge suffered by contracting ℓ.

3 Holomorphic surgery and instanton decay

We first describe informally the ideas behind holomorphic surgery and decay and then give the precise definitions. A *decay* of an instanton is a transformation that lowers the total charge; a *local decay around* ℓ is a transformation that keeps the instanton fixed outside ℓ but lowers the local charge near ℓ, and consequently lowers the global charge as well.

When an instanton is represented by a holomorphic bundle E, then holomorphic surgery provides a precise way to obtain local decay: If the surface Z contains a line ℓ and $N = N(\ell)$ is a small neighbourhood of ℓ in the analytic topology, then lowering the charge of E around ℓ means to replace $E|_N$ by some $E'|_N$ with smaller local charge, while keeping E fixed on $Z-\ell$. The outcome is a new holomorphic bundle which is isomorphic to E over $Z-\ell$, but which has smaller c_2.

Remark 3.1. Note that holomorphic surgery takes one instanton bundle to another instanton bundle; that is, the surgery process keeps $c_1 = 0$, and consequently differs from the more familiar process of elementary transformations (which does not move between instantons).

Definition 3.2. Two instanton bundles E_1 and E_2 on X that are related by holomorphic surgery around ℓ must satisfy $E_1|_{X-\ell} \cong E_2|_{X-\ell}$ holomorphically. If in addition $c_2(E_1) > c_2(E_2)$, we will say that E_1 *decays to* E_2.

We fix a compact surface Z containing a line ℓ with $\ell^2 = -k$, as in the introduction, with $\pi\colon Z \to X$ the contraction of ℓ, and a decomposition $Z = (Z-\ell) \cup N(\ell)$. We will see in Proposition 5.3 that instantons correspond to bundles that are trivial on $Z^o := N(\ell) - \ell = (Z-\ell) \cap N(\ell)$. So, instanton bundles can be given frames on Z^o. We will use theses frames (see definition 3.3 below) in our constructions. Note that (Z_k, Z_k^o) describes the local situation, whereas (Z, Z^o) describes the global situation on a compact manifold. (Although it is not true that any 2-dimensional tubular neighbourhood of a $-k$-line ℓ is biholomorphic to Z_k, it is a consequence of [Ga2] that holomorphic bundles on both $N(\ell)$ and Z_k are completely determined by a finite infinitesimal neighbourhood of ℓ, so that for the purposes of holomorphic surgery and instanton decay, $N(\ell)$ and Z_k can be identified.)

Definition 3.3. Let $\pi_F\colon F \to Z$ be a bundle over Z that is trivial over $Z^o := Z - Y$. Given two pairs $f = (f_1, f_2)\colon Z^o \to \pi_F^{-1}(Z^o)$ and $g = (g_1, g_2)\colon Z^o \to \pi_F^{-1}(Z^o)$ of fibrewise linearly independent holomorphic sections of $F|_{Z^o}$, we say that f is *equivalent* to g (written $f \sim g$) if $\phi := g \circ f^{-1}\colon V|_{Z^o} \to V|_{Z^o}$ extends to a holomorphic map $\phi\colon F \to F$ over the entire Z. A *frame* of F over Z^o is an equivalence class of fibrewise linearly independent holomorphic sections of F over Z^o.

- A *framed* bundle $\bar{E}^f$ on Z is a pair consisting of a bundle $\pi_{\bar{E}}\colon \bar{E} \to Z$ together with a frame of $\bar{E}$ over $Z^o := N(\ell) - \ell$.
- A *framed* bundle V^f on $Z_k := \mathrm{Tot}\big(\mathscr{O}_{\mathbb{P}^1}(-k)\big)$ is a pair consisting of a bundle $\pi_V\colon V \to Z_k$ together with a frame of V over Z_k^o.
- A *framed* bundle E^f on X is a pair consisting of a bundle $\pi_E\colon E \to X$ together with a frame of E over $N(x) - \{x\}$, where $N(x)$ is a small disk neighbourhood of x. We will always take $N(x) := \pi\big(N(\ell)\big)$.

Proposition 3.4. *An isomorphism class* $[\bar{E}^f]$ *of a framed bundle on* Z *is uniquely determined by a pair of isomorphism classes of framed bundles* $[E^f]$ *on* X *and* $[V^f]$ *on* Z_k*. We write* $\bar{E}^f = (E^f, V^f)$*.*

Proof. One needs to observe that any reflexive sheaf on X is completely determined by the complement of the point x, cf. [Ha2, Prop. 1.6]. Hence $E = (\pi_* \bar{E})^{\vee\vee}$ is trivial on $N(x)$ and is completely determined by $E|_{X-\{x\}}$. The rest of the proof is just a verification that the framings have been conveniently defined.

The contraction map gives an isomorphism $i_1 \colon Z - \ell \to X - \{x\}$. Based on [Ga2] we may assume that there is an isomorphism $i_2 \colon N(\ell) \to Z_k$. These induce isomorphisms on the deleted neighbourhoods

$$N(x) - \{x\} \xleftarrow{i_1} Z^o \xrightarrow{i_2} Z^o_k \,.$$

By construction, $\bar{E} = E \sqcup_{(s_1,s_2)=(t_1,t_2)} V$ is made by identifying the bundles as well as the sections over Z^o, so that the bundles satisfy $\bar{E}|_{Z^o} = i_1^* (E|_{N(x)-\{x\}}) = i_2^* (V|_{Z^o_k})$, and the framing (f_1, f_2) of $\bar{E}$ satisfies $(f_1, f_2) = (s_1, s_2) \circ i_1 = (t_1, t_2) \circ i_2$.

Let $\phi \colon E^f \to E'^f$ be an isomorphism such that $\phi \circ (s_1, s_2) = (s'_1, s'_2)$, and let $\xi \colon V^f \to V'^f$ be an isomorphism such that $\xi \circ (t_1, t_2) = (t'_1, t'_2)$. We have the following diagram of bundle maps:

$$\begin{tikzcd}
\bar{E}^f|_{Z-\ell} \arrow[r] \arrow[d, "\pi_{\bar{E}}"'] & E^f \arrow[r, "\phi"] \arrow[d, "\pi_E"'] & E'^f \arrow[d, "\pi_{E'}"] & \bar{E}'^f|_{Z-\ell} \arrow[l] \arrow[d, "\pi_{\bar{E}'}"] \\
Z - \ell \arrow[r, "i_1"'] & X - \{x\} \arrow[r, equal] & X - \{x\} & Z - \ell \arrow[l, "i_1"]
\end{tikzcd}$$

Hence,

$$\bar{E}'^f|_{Z-\ell} = i_1^* (E'^f|_{X-\{x\}}) = i_1^* \circ \phi (E^f|_{X-\{x\}}) = i_1^* \circ \phi \circ i_{1*} (\bar{E}^f|_{Z-\ell}) \,, \tag{3.5}$$

showing that $i_1^* \circ \phi \circ i_{1*}$ is an isomorphism of $\bar{E}$ and $\bar{E}'$ over $Z - \ell$ such that

$$\phi \circ (f_1, f_2) = \phi \circ (s_1, s_2) \circ i_1 = (s'_1, s'_2) \circ i_1 = (f'_1, f'_2) \,. \tag{3.6}$$

On the other hand we have a second diagram of bundle maps:

$$\begin{tikzcd}
\bar{E}^f|_{N(\ell)} \arrow[r] \arrow[d, "\pi_{\bar{E}}"'] & V^f \arrow[r, "\xi"] \arrow[d, "\pi_V"'] & V'^f \arrow[d, "\pi_{V'}"] & \bar{E}'^f|_{N(\ell)} \arrow[l] \arrow[d, "\pi_{\bar{V}'}"] \\
N(\ell) \arrow[r, "i_2"'] & Z_k \arrow[r, equal] & Z_k & N(\ell) \arrow[l, "i_2"]
\end{tikzcd}$$

Therefore,

$$\bar{E}'^f|_{N(\ell)} = i_2^* (V'^f|_{Z_k}) = i_2^* \circ \xi (V^f|_{Z_k}) = i_2^* \circ \xi \circ i_{2*} (\bar{E}^f|_{N(\ell)}) \,, \tag{3.7}$$

showing that $i_2^* \circ \xi \circ i_{2*}$ is an isomorphism of $\bar{E}$ and $\bar{E}'$ over $N(\ell)$ such that

$$\xi \circ (f_1, f_2) = \xi \circ (t_1, t_2) \circ i_2 = (t'_1, t'_2) \circ i_2 = (f'_1, f'_2) \,. \tag{3.8}$$

These isomorphisms agree over the intersection Z^o. In fact, by (3.5) and (3.7),

$$\begin{aligned}
i_1^* \circ \phi \circ i_{1*} (\bar{E}^f|_{Z^o}) = i_1^* \circ \phi (E^f|_{N(x)}) = i_1^* (E'^f|_{N(x)}) \\
= i_2^* (V'^f|_{Z_k - \{0\}}) = i_2^* \circ \xi (V^f|_{Z_k - \{0\}}) = i_2^* \circ \xi \circ i_{2*} (\bar{E}^f|_{Z^o}) \,,
\end{aligned}$$

and moreover they also preserve the framings over the intersection, since over Z^o we have, by (3.6) and (3.8),

$$\phi \circ (f_1, f_2) = (f_1', f_2') = \xi \circ (f_1, f_2)\ .$$

By the gluing lemma this gives an isomorphism over the entire space $\bar{X}$, and we get $\bar{E}' \simeq \bar{E}$. □

Note: Here we have only defined surgery for framed bundles on surfaces. A similar definition of holomorphic surgery can be given in much greater generality for decorated bundles on higher-dimensional varieties; for instance, a broader sense of framing can be used by fixing the isomorphism type of the bundles on a subvariety.

Using (2.1) we can re-state Proposition 3.4 in terms of instantons:

Proposition 3.9. *If* ∇ *and* ∇' *are instantons on* Z*, with* ∇' *obtained from* ∇ *by local decay, then*

$$\text{global charge}(\nabla') = \text{global charge}(\nabla) - \text{local charge}(\nabla, \ell)\ .$$

Proof. Just combine Proposition 3.4 and equality (1.2). □

4 When is total decay near ℓ possible?

Consider the questions: Can every bundle decay totally around ℓ, that is, is every bundle related by holomorphic surgery to a bundle that is trivial around ℓ? Can decay by 1 always happen, that is can every charge n instanton decay locally to charge $n-1$? In the case $k = 1$ the answers are "yes", but for $k > 2$ we will show that the answers to both questions are "no".

In this section we use the well-known concept of *elementary transformations* of Maruyama [M], which we now recall: Let E be a vector bundle over an algebraic variety W. Choose a line bundle L over a Cartier divisor $D \subset W$ and a surjection $r\colon E \to L$ induced by a surjection $\rho\colon E|_D \to L$. Set $E' := \ker(r)$ and $L' := \ker(\rho)$. Since D is a Cartier divisor, E' is a vector bundle on W. By definition E' is called the vector bundle obtained from E by making the *elementary transformation* induced by r, denoted

$$E' = \mathrm{Elm}_L(E)\ .$$

The following diagram, called the *display* of the elementary transformation, clarifies the situation:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & L' \arrow[r] \arrow[u] & E|_D \arrow[r, "\rho"] \arrow[u] & L \arrow[r] & 0 \\
0 \arrow[r] & E' \arrow[r] \arrow[u, "t"] & E \arrow[r, "r"'] \arrow[u] & L \arrow[r] \arrow[u, equal] & 0 \\
 & E(-D) \arrow[r, equal] \arrow[u] & E(-D) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Note that the elementary transformation does not change E outside the divisor, that is, $E|_{W-D} \cong E'|_{W-D}$.

Proposition 4.1. *Let E_1 and E_2 be $\mathfrak{sl}(2,\mathbb{C})$-bundles over Z_k with splitting types j_1 and j_2, respectively. There exists an isomorphism $E_1|_{Z_k^o} \cong E_2|_{Z_k^o}$ if and only if $j_1 \equiv j_2 \mod k$. In particular, E_1 can decay totally over Z_k if and only if $j_1 \equiv 0 \mod k$.*

Proof. We first claim that the bundle $\mathscr{O}_\ell(-k)$ is trivial on Z_k^o. In fact, if $u = 0$ is the equation of ℓ, then $s(z,u) = u$ determines a section of $\mathscr{O}_\ell(-k)$ that does not vanish on Z_k^o.

If a bundle E over Z_k has splitting type j, then by definition, $E|_\ell \cong \mathscr{O}_\ell(-j) \oplus \mathscr{O}_\ell(j)$. So there is a surjection $\rho\colon E|_\ell \to \mathscr{O}_\ell(j)$. The bundle $E' = \mathrm{Elm}_{\mathscr{O}_\ell(j)}(E)$ splits over ℓ as $\mathscr{O}_\ell(-j) \oplus \mathscr{O}_\ell(j+k)$. Therefore we can use the surjection $\rho\colon E'|_\ell \to \mathscr{O}_\ell(j+k)$ to perform a second elementary transformation, and we obtain bundle $E'' = \mathrm{Elm}_{\mathscr{O}_\ell(j+k)}(E')$, which splits over ℓ as $\mathscr{O}_\ell(-j) \oplus \mathscr{O}_\ell(j+2k)$ and has first Chern class $2k$. Tensoring by $\mathscr{O}_\ell(-k)$ we get back to an $\mathfrak{sl}(2,\mathbb{C})$-bundle with splitting type $j+k$. Hence, the transformation

$$\Phi(E) = \otimes \mathscr{O}_\ell(-k) \circ \mathrm{Elm}_{\mathscr{O}_\ell(j+k)} \circ \mathrm{Elm}_{\mathscr{O}_\ell(j)}(E)$$

increases the splitting type by k while keeping the isomorphism type of E over Z_k^o. So we need only to analyse bundles with splitting type $j < k$.

If $j = 0$, the bundle is globally trivial on Z_k. If $j \neq 0$, then $E|_{Z_k^o}$ induces a non-zero element on the fundamental group $\pi_1(Z_k^o) = \mathbb{Z}/k\mathbb{Z}$. □

One interesting consequence of Proposition 4.1 is that instantons do not correspond to bundles whose splitting type does not divide k. Consequently, using the results of the following two sections, we will deduce:

Corollary 4.2. *The self-intersection number of ℓ provides an obstruction to the existence of instantons with certain prescribed numerical invariants.*

In particular, it is not always possible for the local charge to decay by one unless $k = 1$ or 2.

Corollary 4.3. *The self-intersection number of ℓ poses an obstruction to instanton decay.*

Example 4.4. Here some examples, which will be proved below.

1. There is no nontrivial instanton with local charge $\leq k-2$ over the space Z_k when $k > 2$.
2. For $k \geq 2$, there exist $(k-2)$-dimensional families of (unframed) instantons with local charge $k-1$ over Z_k.

5 Existence of instantons

In [Ga2] it is shown that every holomorphic bundle on Z_k is an algebraic extension of line bundles. It then follows that any rank-2 bundle E over Z_k with $c_1(E) = 0$ is an extension of the form

$$0 \to \mathscr{O}(-j) \to E \to \mathscr{O}(j) \to 0\,.$$

Thus existence of moduli $\mathscr{M}_j(k)$ of bundles with any splitting type j over Z_k is an immediate consequence of the fact that $\mathrm{Ext}^1\big(\mathscr{O}_{Z_k}(j), \mathscr{O}_{Z_k}(-j)\big) \neq \varnothing$. Moreover, in [BGK, Theorem 4.2] it is shown that for $j > k$

$$\dim \mathscr{M}_j(k) = 2j - k - 2\,. \tag{5.1}$$

Note that $\dim \mathscr{M}_j(k)$ is not the dimension of $\mathrm{Ext}^1\big(\mathscr{O}_{Z_k}(j), \mathscr{O}_{Z_k}(-j)\big)$ as a vector space, since bundle isomorphisms impose several equivalences; rather it is the dimension of the dense, open stratum of $\mathscr{M}_j(k)$ seen as a variety. Now we analyse which of these bundles correspond to instantons. Firstly, we look at them from the point of view of decay, and secondly from the point of view of differential geometry.

The *energy* of an instanton on X is given by $\frac{1}{2g^2}\int_X \|F\|^2$. The minimum possible energy of the instanton is bounded from below by the total charge of the instanton. To see this, note the following inequalities:

$$0 \le \int_X \|F \pm *F\|^2 = 2\int_X \|F \wedge *F \pm F \wedge F\|$$

Therefore,

$$\frac{1}{2g^2}\int_X \|F\|^2 = \frac{1}{2g^2}\int_X \|F \wedge *F\| \ge \frac{1}{2g^2}\left|\int_X F \wedge F\right| = \frac{8\pi^2 |n|}{g^2}\,.$$

Consequently, the probability of finding an instanton with $c_2 = n$ is $\propto e^{-n}$. Arguing from the physical point of view, since systems always tend to go to their lowest energy state, an instanton with a high charge will prefer to decay to an instanton of a lower charge unless there is some obstruction to its decay. It is reasonable to expect that this behaviour holds locally as well. Thus it should be possible to lower a local charge by a local transformation. Instanton bundles, therefore, ought to allow for full local decay; combining with Proposition 4.1 this implies that only bundles which are trivial on Z_k^o can correspond to instantons. Moreover, the finite-energy condition for instantons, *viz.* $\int_X \|F\|^2 \le \infty$, requires that $F \to 0$ at infinity, and accordingly an instanton bundle E on Z_k should be trivial and trivialised at infinity. This requirement in turn fixes boundary conditions and guarantees that instantons have good gluing properties.

Mathematically, the correct way to decide which bundles correspond to instantons is to go through the Kobayashi–Hitchin correspondence (cf. [LT]). A unitary, anti-self-dual connection ∇ on a smooth bundle E decomposes as $\nabla = \partial + \bar\partial$, where $\bar\partial$ is considered as a holomorphic structure on E; and the Kobayashi–Hitchin correspondence claims that the map $\nabla \mapsto \bar\partial$ is invertible. In the compact case, this correspondence was proved by Donaldson [D1] for projective algebraic surfaces, by Uhlenbeck and Yau [UY] for Kähler surfaces, and by Buchdahl [Bu] for surfaces with a Gauduchon metric. In the non-compact case, this correspondence was proved by Donaldson [D2] for $\mathbb{C}^2$ and by King [Kn] for $\mathbb{C}^2$ blown up at the origin, which in this paper is denoted by Z_1. In the former, instantons on $\mathbb{C}^2$ are identified with instantons on $\mathbb{C}P^2$ framed at a line at infinity; and in the latter framed instantons on Z_1 are identified with instantons on the first Hirzebruch surface Σ_1 which are trivial on the line at infinity. As in the non-compact cases of $\mathbb{C}^2$ and Z_1 we identify framed instantons on Z_k with instantons on the k^{th} Hirzebruch surface $\Sigma_k := \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^1}(k) \oplus \mathscr{O}_{\mathbb{P}^1}\big)$ trivialised on the line at infinity.

LeBrun [LB1] provided metrics over the spaces Z_k that are well suited for instanton problems. He showed that Z_k admits a complete, zero scalar curvature, asymptotically flat Kähler metric g. In particular, this metric is anti-self-dual. Moreover, he showed that up to multiplication by an overall constant, there is exactly one such metric g which is $SU(2)$-invariant. Using this asymptotically flat metric, triviality at the line at infinity still seems a natural condition to impose. In fact, consider the orbifold $\bar Z_k$ obtained from Z_k by adding one point at infinity, or equivalently, obtained from Σ_k by contracting the line at infinity ℓ_∞. Then it follows from [LB2, p. 235] that $\bar Z_k$ is an ASD conformal orbifold compactification of Z_k. This orbifold compactification of Z_k has an orbifold twistor space W, cf. [LB3]. The complex structure on Z_k yields a complex surface in W, and adding the orbifold twistor line at infinity compactifies this surface to the Hirzebruch surface Σ_k. Using the Ward correspondence together with Uhlenbeck's removable singularities theorem [U], an L^2-instanton on Z_k gives rise to a holomorphic bundle on Σ_k that is trivial on ℓ_∞. So, once again, from this second point of view we arrive at the conclusion that instanton bundles on Z_k should be the ones that are trivial at infinity. We set the *ad hoc* definition of *framed stability*, cf. Definition 3.3.

Definition 5.2. A rank-2 bundle over Z_k is called *framed-stable* if it is holomorphically trivial and framed on Z_k^o.

This allows a statement the *Kobayashi–Hitchin correspondence* on Z_k:

Proposition 5.3. *There exists a one-to-one correspondence between framed $SU(2)$-instantons on Z_k with local charge n and framed-stable $\mathfrak{sl}(2,\mathbb{C})$-bundles on Z_k with $\chi^{\mathrm{loc}} = n$.*

Schematically,

$$\left\{ \begin{array}{c} SU(2)\text{-instantons on} \\ Z_k \text{ with local charge } n \end{array} \right\} \overset{\text{K.–H.}}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{stable } SL(2)\text{-bundles} \\ \text{on } Z_k \text{ with } \chi^{\mathrm{loc}} = n \end{array} \right\}. \tag{5.4}$$

Corollary 5.5. *An $\mathfrak{sl}(2,\mathbb{C})$-bundle over Z_k represents an instanton if and only if its splitting type is a multiple of k.*

Proof. By definition an instanton bundle must be trivial at infinity, now apply Proposition 4.1. ☐

In particular, note that any bundle on Z_k with splitting type $j \not\equiv 0 \mod k$ does not correspond to an instanton; the physical interpretation of such a bundle does not seem to be known.

6 Gaps in local charges and moduli

In this section we study gaps in local charges. We show that not all numerically admissible values of χ occur for instanton bundles when $k > 2$. Recall that by Definition 2.3 the local charge is the sum of the width and the height: $\chi^{\mathrm{loc}}(E) = \mathbf{w}(E) + \mathbf{h}(E) = h^0\big(X; (\pi_* E)^{\vee\vee}/\pi_* E\big) + h^0\big(X; R^1\pi_* E\big)$.

6.1 Direct computation of instanton widths

Results in this section depend on a number of "direct calculations". We explain briefly how those are carried out and provide an open computer implementation. We outline the computer algorithm, following closely the ideas in [GaS] and keeping to minimal detail. The implementation of the algorithm can be obtained from `http://www.maths.ed.ac.uk/~s0571100/Instanton/`. Our language of choice is *Macaulay 2* for its native support of high-level concepts of commutative algebra (such as modules, generators, cokernels); though conceivably a different computer algebra software may be used.

Let $E \to Z_k$ be a holomorphic rank-2 vector bundle over the complex surface $Z_k = \mathrm{Tot}\big(\mathcal{O}(-k)\big)$ with $c_1(E) = 0$. The canonical coordinates on $Z_k = U \cup V$ are $U = \{z, u\}$ and $V = \{\xi, v\}$ such that $\xi = z^{-1}$ and $v = z^k u$. A holomorphic bundle E on Z_k is algebraic [Ga2]. If the splitting type of E is j, then it can be expressed by a canonical transition function

$$T = \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix},$$

where

$$p(z,u) = \sum_{r=1}^{\left\lfloor \frac{2j-2}{k} \right\rfloor} \sum_{s=kr-j+1}^{j-1} p_{rs}\, u^r z^s\ . \tag{6.1}$$

The computation of the instanton width is now equivalent to the computation of the dimension of the cokernel of the natural evaluation map $M \hookrightarrow M^{\vee\vee}$, where M is a module that is related to the space of holomorphic sections of E: Let (a,b) be a generic section of E given over the (z,u)-chart by functions $a, b \in \mathbb{C}[[z,u]]$. This implies that on the other chart, the local section

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} z^j\, a + p\, b \\ z^{-j}\, b \end{pmatrix}$$

is holomorphic in $(z^{-1}, z^k\, u)$. Writing $a(z,u) = a_{rs} u^r z^s$ and $b(z,u) = b_{rs} u^r z^s$, this means that for each fixed index r only a finite number of a_{rs} and b_{rs} can be non-zero, and there are relations between the non-zero coefficients of a and b (unless $p \equiv 0$). The space of sections of E is thus generated by terms $(u^r z^s, u^{r'} z^{s'})$, of which most are multiples or linear combinations of a finite set of true generators.

For our computation we need to consider generators and relations after contracting the zero-section of Z_k to a point, i.e. on the direct image under the contracting map $\pi\colon Z_k \to X_k$. Here X_k is the singular surface (smooth only for $k = 1$) given in coordinates by $S = \{x_0, \ldots, x_k\}/(x_i x_{i+t} - x_{i+1} x_{i+t-1})$ with $0 \le i \le i + t \le k + 1$, and the map π is given by $x_i = z^i u$. The module M is now the space of sections of E *when the relations are expressed in terms of the x_i downstairs*, i.e. as an S-module.

Example 6.2. For the first two values of k we have $Z_1 \to X_1 = \mathbb{C}^2 = \{x_0, x_1\}$ and $Z_2 \to X_2 = \{x_0, x_1, x_2\}/(x_0 x_2 - x_1^2)$, where $u \mapsto x_0$, $zu \mapsto x_1$ and $z^2 u \mapsto x_2$.

The first two generators coming from b_{00} and b_{01} are $\beta_0 = (0,1)$ and $\beta_1 = (0,z)$, respectively, and they are related over S by

$$\begin{aligned} x_1 \beta_0 &= x_0 \beta_1 && \text{(on } Z_1 \text{ and } Z_2\text{), and} \\ x_2 \beta_0 &= x_1 \beta_1 && \text{(on } Z_2 \text{ only).} \end{aligned}$$

The concrete case $j = k$ and $p(z,u) = zu$ is worked out in the proof of Theorem 6.8.

Computer algorithm The automatic computation of the instanton width of a bundle E with $c_1(E) = 0$, splitting type j and extension class p can now proceed in several stages:

1. (Optional) The extension class p may possibly be reduced to a smaller, cohomologous class p' by truncating terms according to (6.1), but care needs to be taken when $u \nmid p$. This step is only useful to optimise computation time, it is not necessary for the algorithm to work.

2. Define a generic section (a,b) with $a(z,u) = \sum_{r,s} a_{rs} u^r z^s$ and $b(z,u) = \sum_{r,s} b_{rs} u^r z^s$. There exist bounds on r above which all generators corresponding to the $a_r s$ and $b_r s$ terms are guaranteed to be multiples (over S) of the lower generators, so these are genuine polynomials. Moreover, b can and must be chosen so that $z^{-j}\, b$ is holomorphic in z^{-1} and $z^k u$.

3. Compute the first coordinate of the section on the second chart: $f = z^j a + p\,u$. Now for each term $z^s u^r$, whenever $s > kr$ the coefficient must vanish; this gives relations between the coefficients.

4. The module M is now built up step by step by substituting the relations back into a and b, and setting one coefficient to 1 and all others to 0 (call the result a_1, b_1 just for now), transforming (a_1, b_1) into expressions over S and adding the resulting vector as a generator of M. After doing this for all coefficients, a presentation for M as a module over S is obtained. (See [GaS] for details of this construction, in particular how it deals with "fake relations".)

5. The computation of the cokernel of ev: $M \hookrightarrow M^{\vee\vee}$ relies on [GaS, Lemma 2.1] and can be done very easily in *Macaulay 2*. The instanton width of E is the dimension of coker(ev) as a $\mathbb{C}$-vector space.

6.2 Computation of instanton heights

We will use the following formula for the height, which is proved in [BGK]:

Theorem 6.3 ([BGK, 2.6])**.** *Let E be a non-split bundle represented in canonical form by (j,p), and let $m > 0$ be the smallest exponent of u appearing in p. With $\mu = \min\big(m, \lfloor\frac{j-2}{k}\rfloor\big)$, we have*

$$l(R^1\pi_* E) = \mu\left(j - 1 - k\,\frac{\mu-1}{2}\right)\,, \tag{6.4}$$

and equality holds if p is holomorphic on Z_k.

6.3 Results

Lemma 6.5. *Let E_j be a rank-2 bundle over Z_k with $c_1 = 0$ and splitting type $j < k$. Then*

$$\chi^{\mathrm{loc}}(E_j) = j - 1\,. \tag{6.6}$$

Proof. By [Ga2, Theorem 3.3] it follows that if $j < k$ then $E_j \cong \mathcal{O}_{Z_k}(j) \oplus \mathcal{O}_{Z_k}(-j)$. By definition, $\chi^{\mathrm{loc}}(E_j) = \mathbf{w}(E_j) + \mathbf{h}(E_j)$. Direct computation (see [BGK]) then shows that $\mathbf{w}(E_j) = 0$ and $\mathbf{h}(E_j) = j - 1$. □

Proposition 6.7. *Let E be a rank-2 reflexive sheaf on Z_k with splitting type $\geq k$, then $\chi\big(x, E\big) \geq k - 1$.*

Proof. By semi-continuity of $\chi\big(x,E\big)$ on the splitting type, the lowest value of $\chi\big(x,E\big)$ must occur for splitting type k. By definition,

$$\chi\big(x,E\big) \geq h^0\big(X; R^1\pi_* E\big)\,,$$

and now apply the formula (6.4). □

Theorem 6.8. *The minimal local charge of a nontrivial $SU(2)$-instanton on Z_k is $\chi_k^{\min} = k-1$. The local moduli space of (unframed) instantons on Z_k having fixed local charge $\chi_k^{\min}$ has dimension $k-2$.*

Proof. By Corollary 5.5, a nontrivial instanton bundle over Z_k has splitting type kn for some $n \in \mathbb{Z}$, $n > 0$. Hence, the smallest nontrivial splitting type is exactly k, and the generic such instanton corresponds to an element of $\operatorname{Ext}^1_{Z_k}\big(\mathcal{O}(k), \mathcal{O}(-k)\big)$ which is nontrivial on the first formal neighbourhood.

The dimension of the local moduli space with fixed $\chi_k^{\min}$ is obtained from formula (5.1) setting $j = k$. We compute χ for the bundle E corresponding to the extension class $[zu] \in \operatorname{Ext}^1\big(\mathcal{O}(k), \mathcal{O}(-k)\big)$.

By Equation (6.4), $\mathbf{h}(E) = k - 1$.

To compute $\mathbf{w}(E) = h^0\big(X, (\pi_* E)^{\vee\vee}/\pi_* E\big)$, we use the method described in Section 6.1: Let Q be the skyscraper sheaf defined by the exact sequence

$$0 \longrightarrow \pi_* E \longrightarrow \big(\pi_* E\big)^{\vee\vee} \longrightarrow Q \longrightarrow 0\,.$$

Then $\mathbf{w}(E)$ equals the dimension of $Q_x^\wedge$ as a k_x-module. So, we need to study the map $\pi_* E_x^\wedge \to (\pi_* E_x^\wedge)^{\vee\vee}$ and compute the dimension of the cokernel, i.e. we need to compute the module structure on $M := \pi_* E_x^\wedge$ and study the natural map $M \hookrightarrow M^{\vee\vee}$. By the Theorem on Formal Functions (see [Ha1, p. 277]),

$$M \cong \varprojlim_n H^0\big(\ell_n; E|_{\ell_n}\big)\,,$$

where ℓ_n is the n^{th} infinitesimal neighbourhood of ℓ. Since the extension class has degree 1 in u, then for $n \geq 1$,

$$H^0\big(\ell_n; E|_{\ell_n}\big) \cong H^0\big(\ell_1; E|_{\ell_1}\big)\,.$$

Therefore, the inverse limit stabilises at 1, giving $M \cong H^0\big(\ell_1; E|_{\ell_1}\big)$.

To compute the generators of M, we write the transition matrix for E explicitly. We set $Z_k = U \cup V$, where $U \cong \mathbb{C}^2 \cong V$, with change of coordinates $U \ni (z, u) \mapsto (z^{-1}, z^k u) \in V$ on $U \cap V \cong \mathbb{C} - \{0\} \times \mathbb{C}$. Then in these coordinate charts, E is given by transition matrix

$$T = \begin{pmatrix} z^k & zu \\ 0 & z^{-k} \end{pmatrix}.$$

Set $\alpha = \left(\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right)$ and $\beta_i = \left(\begin{smallmatrix} 0 \\ z^i \end{smallmatrix}\right)$ for $i = 0, \ldots, k-1$, $\beta_k = \left(\begin{smallmatrix} -zu \\ z^k \end{smallmatrix}\right)$. Then a presentation for M is given by $M = \langle \alpha, \beta_0, \ldots, \beta_k \rangle / R$, where R is the set of relations $\beta_i x_0 - \beta_{i-1} x_1 = 0$ for $i = 1, \ldots, k-1$ and $\beta_k x_0 - \beta_{ik-1} x_1 - \alpha x_1 = 0$. Standard computations (which can be performed either by hand, or with a computer algebra program) then show that the evaluation map $\rho\colon M \to M^{\vee\vee}$ is surjective. Hence $\mathbf{w}(E) = 0$.

Summing up, $\chi_k^{\min} = \chi^{\text{loc}}\big(E\big) = \mathbf{w}(E) + \mathbf{h}(E) = 0 + (k-1)$. □

Remark 6.9 (Gaps in local instanton charges). The non-existence of instantons with certain local charges on the spaces Z_k when $k > 2$ is in stark contrast with what happens in the case $k = 1$. In fact, by [BG1, Theorem 0.2], for every non-negative integer n there exist instantons on Z_1 with local charge n. More precisely, by [BG1, Theorem 0.2], for every pair of integers (w, h) satisfying $j - 1 - e \leq w \leq j(j-1)/2 - j\,e$ and $1 \leq h \leq j(j+1)/2$ with $j \geq 0$ and $e \geq 0$ or -1, there exists a rank-2 vector bundle E on Z_1 with splitting type $(j, -j + e)$ having $\mathbf{w}(E) = w$ and $\mathbf{h}(E) = h$. Hence, there are no gaps in local charges for instantons over Z_1.

References

[BG1] Ballico, E. and Gasparim, E., *Vector bundles on a neighborhood of a curve in a surface and elementary transformations*, Forum Math. **15** no. 1, 115–122 (2003).

[BGK] Ballico, E. Gasparim, E. and Köppe, T., *Vector bundles near a negative curve: moduli and local Euler characteristic.* Preprint.

[Bl] Blache, R., *Chern classes and Hirzebruch–Riemann–Roch theorem for coherent sheaves on complex-projective orbifolds with isolated singularities*, Math. Z. **222** no. 1, 7–57 (1996).

[Bu] Buchdahl, N. P., *Hermitian–Einstein connections and stable vector bundles over compact algebraic surfaces*, Math. Ann. **280**, 625–648 (1988).

[BHMM] Boyer, C. P., Hurtubise, J. C., Mann, B. M. and Milgram, R. J., *The topology of instanton moduli spaces. I. The Atiyah–Jones conjecture.* Ann. of Math. (2) **137** no. 3, 561–609 (1993).

[D1] Donaldson, S. K., *Instantons and geometric invariant theory*, Comm. Math. Phys. **93**, 453–460 (1984).

[D2] Donaldson, S. K., *Anti-self-dual connections over complex algebraic surfaces and stable vector bundles*, Proc. Lond. Math. Soc. (3) **50**, 1–26 (1985).

[Ga1] Gasparim, E., *The Atiyah–Jones conjecture for rational surfaces.* To appear in Advances Math. (2008).

[Ga2] Gasparim, E., *Holomorphic bundles on $\mathscr{O}(-k)$ are algebraic*, Comm. Algebra **25** no. 10, 3001–3009 (1997).

[GaS] Gasparim, E. and Swanson, I., *Computing instanton numbers of curve singularities*, J. Symbolic Comput. **40** no. 2, 965–978 (2005)

[GO] Gasparim, E. and Ontaneda, P., *Three applications of instanton numbers*, Comm. Math. Phys. **270** no.1, 1–12 (2007).

[Ha1] Hartshorne, R., *Algebraic geometry.* Graduate Texts in Mathematics **52**. Springer-Verlag, New York-Heidelberg (1977).

[Ha2] Hartshorne, R., *Stable reflexive sheaves*, Math. Ann. **254** no. 2, 121–176 (1980).

[HM] Hurtubise, J. and Milgram, R. J., *The Atiyah–Jones conjecture for ruled surfaces*, J. Reine Angew. Math. **466**, 111–143 (1995).

[Kn] King, A., *Instantons and holomorphic bundles on the blown-up plane*, D. Phil. Thesis, Worcester College, Oxford (1998).

[LB1] LeBrun, C., *Counter-examples to the generalized positive action conjecture*, Commun. Math. Phys. **118**, 591–596 (1988).

[LB2] LeBrun, C., *Explicit self-dual metrics on $\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2$*, J. Differential Geom. **34** no. 1, 223–253 (1991).

[LB3] LeBrun, C., *Twistors, Kähler manifolds, and bimeromorphic geometry. I.* J. Amer. Math. Soc. **5** no. 2, 289–316 (1992).

[LT] Lübke, M. and Teleman, A., *The Kobayashi–Hitchin correspondence.* World Scientific Publishing Co., Inc., River Edge, NJ (1997).

[M] Maruyama, M., *Elementary transformations in the theory of algebraic vector bundles*, Algebraic geometry (La Rábida, 1981), 241–266, Lecture Notes in Math. **961**, Springer, Berlin, 1982.

[U] Uhlenbeck, K. *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** no. 1, 11–29 (1982).

[UY] Uhlenbeck, K. and Yau, S. T., *On the existence of Hermitian–Yang–Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39**, suppl. S257–S293 (1986).

Elizabeth Gasparim, Thomas Köppe
School of Mathematics, The University of Edinburgh
James Clerk Maxwell Building, The King's Buildings
Mayfield Road, Edinburgh, UK, EH9 3JZ
E-mail: `Elizabeth.Gasparim@ed.ac.uk`
E-mail: `t.koeppe@ed.ac.uk`

Pushan Majumdar
Institut für Theoretische Physik
Westfälische Wilhelms-Universität Münster
48149 Münster, Germany
E-mail: `pushan@uni-muenster.de`