Expander ℓ_0-Decoding

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- For $x \in \mathbb{R}^n$, let $\mathrm{supp}(x) = \{i \in [n] : x_i \neq 0\}$.
- Define $\|x\|_0 := |\,\mathrm{supp}(x)|$ and let

$$\chi_k^n := \{x \in \mathbb{R}^n : \|x\|_0 \leq k\},$$

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- In CCS A is set to be an *expander matrix*.

Expander matrices

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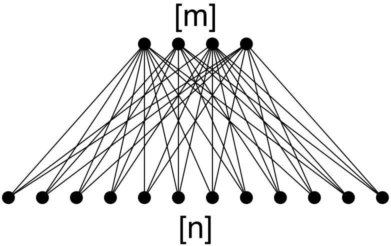

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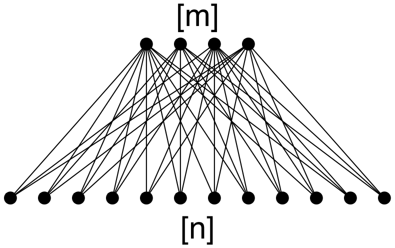

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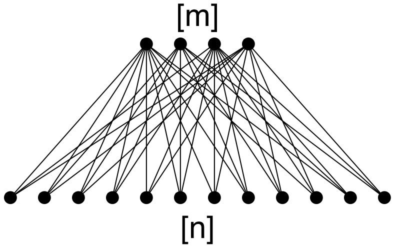

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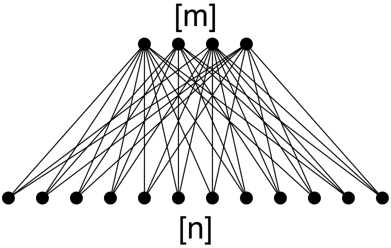

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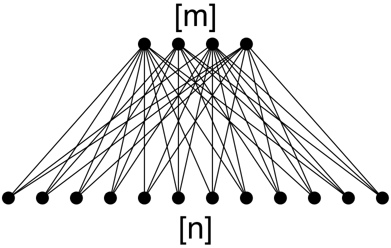

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$A \in \mathbb{R}^{m \times n}$ is a sparse binary matrix with $d << m$ ones per column (we say that $A \in \mathbb{E}_{k,\varepsilon,d}$).

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Expander	$\mathcal{O}(dn)$	$\mathcal{O}(dn)$	$\mathcal{O}(dn)$	$\mathcal{O}(k\log(n/k))$

Iterative Greedy Algorithms

Algorithm: Iterative greedy CCS algorithms

Data: $A \in \mathbb{R}^{m \times n} \cap \mathbb{E}_{k,\varepsilon,d}$; $y \in \mathbb{R}^m$
Result: $\hat{x} \in \mathbb{R}^n$ s.t. $y = A\hat{x}$
$\hat{x} \leftarrow 0$, $r \leftarrow y$;
while *not converged* **do**
  Compute a score s_j and an update u_j $\forall j \in [n]$;
  Select $S \subset [n]$ based on a rule on s_j;
  $\hat{x}_j \leftarrow \hat{x}_j + u_j$ for $j \in S$;
  k-threshold $\hat{x}$;
  $r \leftarrow y - A\hat{x}$;
end

Iterative Greedy Algorithms

Algorithm	Objective	s_j	Complexity
SMP [1]	ℓ_1	median$(r_{\mathcal{N}(j)})$	$\mathcal{O}((nd + n\log n)\log \|x\|_1)$
SSMP [2]	ℓ_1	median$(r_{\mathcal{N}(j)})$	$\mathcal{O}((\frac{d^3 n}{m} + n)k + (n\log n)\log \|x\|_1)$
LDDSR [3] / ER [4]	ℓ_0	mode$(r_{\mathcal{N}(j)})$	$\mathcal{O}((\frac{d^3 n}{m} + n)k)$
Serial-ℓ_0 [5]	ℓ_0	$\|r\|_0$-decrease	$\mathcal{O}(dn\log k)$
Parallel-ℓ_0 [5]	ℓ_0	$\|r\|_0$-decrease	$\mathcal{O}(dn\log k)$

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The theoretical guarantees of our algorithms require additional structure on x.

Signal model

Definition (Dissociated signals)

A signal $x \in \chi_k^n$ is dissociated if

$$\sum_{j \in T_1} x_j \neq \sum_{j \in T_2} x_j \quad \forall\ T_1, T_2 \subset \mathrm{supp}(x) \text{ with } T_1 \neq T_2 \tag{2}$$

Serial-ℓ_0

Algorithm: Serial-ℓ_0 [5]

Data: $A \in \mathbb{R}^{m \times n}$; $y \in \mathbb{R}^m$; $\alpha \in [d]$
Result: $\hat{x} \in \mathbb{R}^n$ s.t. $y = A\hat{x}$

```
x̂ ← 0, r ← y;
while not converged do
    for j ∈ [n] do
        T ∈ {ω_j ∈ ℝ : ‖r‖_0 − ‖r − ω_j a_j‖_0 > α};
        for ω_j ∈ T do
            x̂_j ← x̂_j + ω_j;
        end
    end
    r ← y − Ax̂;
end
```

Parallel-ℓ_0

Algorithm: Parallel-ℓ_0 [5]

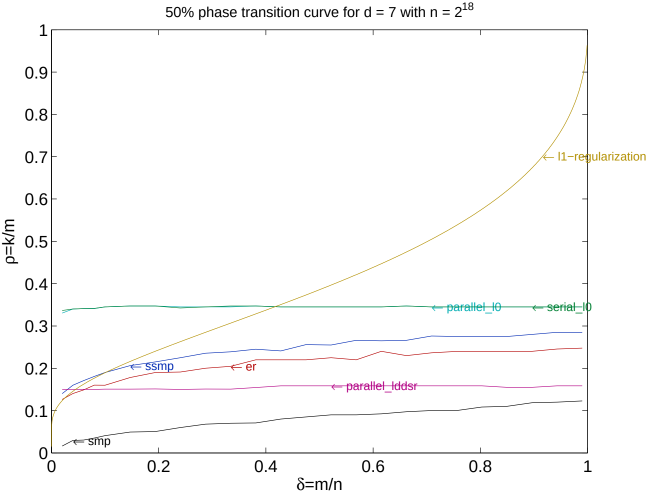
Data: $A \in \mathbb{R}^{m \times n}$; $y \in \mathbb{R}^m$; $\alpha \in [d]$
Result: $\hat{x} \in \mathbb{R}^n$ s.t. $y = A\hat{x}$
$\hat{x} \leftarrow 0$, $r \leftarrow y$;
while *not converged* **do**
    $T \leftarrow \{(j, \omega_j) \in [n] \times \mathbb{R} : \|r\|_0 - \|r - \omega_j a_j\|_0 > \alpha\}$;
    for $(j, \omega_j) \in T$ **do**
        $\hat{x}_j \leftarrow \hat{x}_j + \omega_j$;
    end
    $r \leftarrow y - A\hat{x}$;
end

Exp-ℓ_0-De: Theoretical guarantees

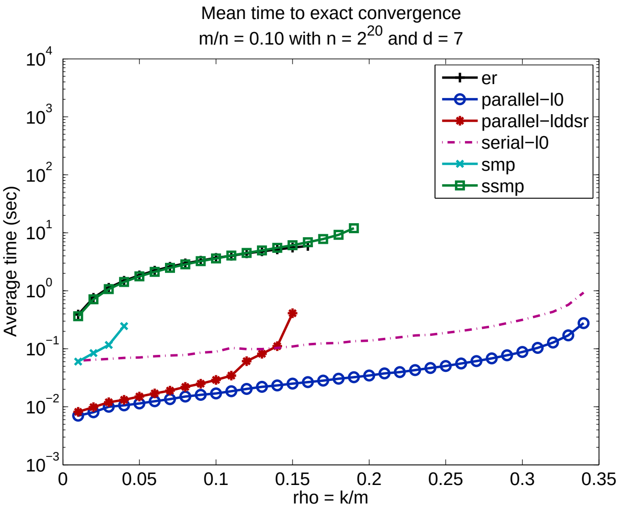
Theorem

Let $A \in \mathbb{E}_{k,\varepsilon,d} \cap \mathbb{R}^{m \times n}$*, and* $x \in \mathbb{R}^n$ *be a* k*-sparse dissociated signal. If* $\varepsilon < 1/4$ *and* $\alpha = d/2$*, then Serial-*ℓ_0 *and Parallel-*ℓ_0 *solve* $y = Ax$ *in* $\mathcal{O}(dn \log k)$ *operations.*

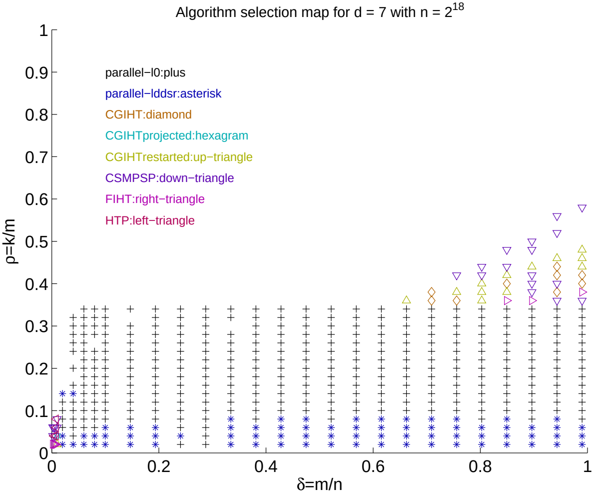
Improved phase transition



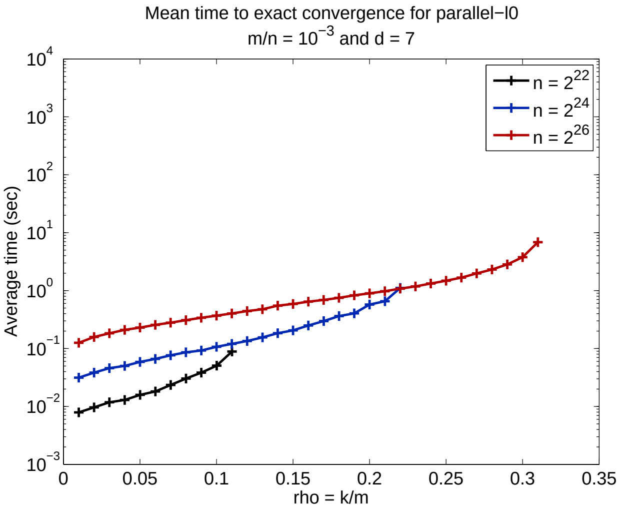
Fastest algorithm in CCS



Fastest algorithm in CS for dissociated x



High phase transition $\delta \approx 0$



The end :)

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