Low-regularity well-posedness for semilinear wave equations.

see Tao: Nonlinear dispersive equations.

Smith-Tataru: sharp LWP results for the ~~—~~ NLW eqn.

Example: $\Box = -\partial_t^2 + \Delta_{\mathbb{R}^{n+1}}$

$$(\text{IVP}) \begin{cases} (0) \quad \Box u = \pm |u|^{p-1} u & u : \mathbb{R}^{n+1} \to \mathbb{R} \\ u(0,x) = u_0(x) \in H^s \\ \partial_t u(0,x) = u_1(x) \in H^{s-1} \end{cases}$$

with $s-1 < \frac{n}{2}$ (Sobolev gives L^∞ by $H^{n/2+\varepsilon}$, not bdd, certainly not smooth)

(Recall $\|u\|_{\dot{H}^s}^2 = \int |\xi|^{2s} |\hat{u}|^2 d\xi = \sum_{|\alpha|=s} \int |\partial^\alpha u|^2 dx$ if $s \in \mathbb{N}$,

$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \|u\|_{\dot{H}^s}^2$.)

(Semilinear: ~~right-hand side depends on~~ $\Box u = F(u)$,)

(System for $\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$ with $\vec{u}_0 \in H^s \times H^{s-1}$.)

Scaling

If u solves (0), then so does

$$u^\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$$

and $\|u^\lambda\|_{\dot{H}^s} = \lambda^{\frac{n}{2} - s - \frac{2}{p-1}} \|u\|_{\dot{H}^s}$

Definition (0) is s_c-critical if

$$\frac{n}{2} - s_c = \frac{2}{p-1}.$$

If $0 \le s < s_c$, then as $\lambda \to 0$, $\|u\|_{H^s} \to 0$

and, if u blows up at time T^*, u^λ blows up at time $\lambda T^* = 0$.

$\Rightarrow \exists u \in H^s$ with no solution.

(Future domain of dependence
Sum pieces
Even if no blow-up, still ill-posed)

Duhamel's formula

Let $S(t,t')$ be the evolution flow

so that $S(t,t')\vec{u}_0 = \vec{u}(t)$ where $\Box\vec{u} = 0$, $\vec{u}(t') = \vec{u}_0$.

If $\vec{u}$ solves (IVP)

$$(1) \qquad \begin{cases} \Box\vec{u} = \vec{N}(\vec{u}) \\ \vec{u}(0,x) = \vec{u}_0(x) \end{cases}$$

then $\vec{u}$ satisfies

$$(2) \qquad \vec{u}(t) = S(t,0)\vec{u}_0 + \int_0^t S(t,t')\,\vec{N}(\vec{u}(t'))\,dt'.$$

Solutions

Def'n ~~A~~

Keep, why not?

~~A classical solution to (1) is a $u \in C^\infty(\mathbb{R}^{n+1})$ satisfying (1).~~

KEEP

~~A distributional solution to (1) is a~~ distribution satisfying (2).

A strong solution to (1) on an interval I is a distributional solution with $u \in C^0_t(I, H^s \times H^{s-1})$.

~~A weak solution to (1) on an interval I is a distributional solution, $u \in L^\infty_t(I, H^s \times H^{s-1})$~~

(Strong are important. Allow low regularity, but if C^∞ init data, get classical solutions.

(weak & Navier-Stokes).

Def'n:

The IVP (1) is locally well-posed (LWP) in H if (a function space on $\mathbb{R}^n$, H,)

there's a function space on $\mathbb{R}^{n+1}$, Y,

$\forall u_0^* \in H$:

$\exists T>0$, open nbhd B of u_0^* in H: time of existence dep on solution

$\forall \vec{u}_0 \in B$:

$\exists \vec{u} \in C_t^0([0,T], H)$: such that — solution for u_0^* & friends upto T.

(1- existence) $\vec{u}$ is a strong solution to (1) with data $\vec{u}_0$.

(2- uniqueness) $\vec{u}$ is the unique solution in $C_t^0([0,T],H) \cap Y$. Y gives uniqueness

(3- continuous in data) The map $u_0 \mapsto u : B \to C_t^0([0,T],H)$ is continuous.

It is globally well-posed if $T=\infty$.

It has unconditional uniqueness if $Y = C_t^0([0,T],H)$

It has norm-dependent time of existence if $T = T(\|u_0^*\|_H)$.

(Recall

~~Thm [Strichartz]~~ wave s-admissible (q,r): $\quad \frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s$.

Thm [Strichartz]

If $n \geq 2$, (q_1,r_1), (q_2,r_2) wave s-admissible

$$\Box u = F \quad \text{in } [0,T] \times \mathbb{R}^n$$
$$\vec{u} = \vec{u}_0$$

then
$$\|\vec{u}\|_{C_t^0(H_s \times H^{s-1})} + \|u\|_{L_t^{q_1}([0,T], L_x^{r_1})} \lesssim \|\vec{u}_0\|_{H^s \times H^{s-1}} + \|F\|_{L^{q_2'}([0,T], L_x^{r_2'})} ,$$
)

Thm

$\Box u = \pm |u|^2 u$ is LWP in $H^s \times H^{s-1}$ with $s \geq s_c$.

pf :

Note $q = r = 4$ is $s_c = \frac{1}{2}$ admissible

(3)

$$\boxed{\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s_c = \frac{2}{p-1}}$$

$$\frac{1+3}{4} = \frac{3}{2} - \frac{1}{2} = \frac{2}{3-1} = 1.$$

(2) Idea: contraction in $Y = L^4_{tx}$.

(1) Use $L^q_t L^r_x$ for $L^q_t([0,T], L^r_x)$ with T to be chosen later.
$L^q_{tx} = L^q_t L^q_x$.

Let $u^0 = 0$ Picard iteration.

$$u^i = S(t,t')\, u^0 + \Phi[u^{i-1}]$$

equivalently u^i solves $\square u^i = N(u^{i-1})$

$$\vec{u}^{\,i}(0) = \vec{u_0}.$$

and $u^i - u^{i-1}$ solves $\square\,(u^i - u^{i-1}) = N(u^{i-1}) - N(u^{i-2})$

$$(\vec{u}^{\,i} - \vec{u}^{\,i-1})(0) = \vec{0}.$$

By Strichartz (with $\tilde{q} = \tilde{r} = q = r = 4$)

$$\| u^i - u^{i-1} \|_{L^4_{tx}} \lesssim \| \vec{0} \|_{H^s \times H^{s-1}} + \| N(u^{i-1}) - N(u^{i-2}) \|_{L^{4/3}_{tx}}$$

$$\lesssim \| (u^{i-1})^3 - (u^{i-2})^3 \|_{L^{4/3}_{tx}}$$

$$\lesssim \| \left((u^{i-1})^2 + (u^{i-2})^2\right) (u^{i-1} - u^{i-2}) \|_{L^{4/3}_{tx}}$$

Apply Hölder with $3/2$ vs 3

$$< C \left(\| u^{i-1} \|^2_{L^4_{tx}} + \| u^{i-2} \|^2_{L^4_{tx}} \right) \| u^{i-1} - u^{i-2} \|_{L^4_{tx}}.$$

If $\forall i < n$, $\| u^i \|_X < \frac{1}{2\sqrt{2}}$, then $\| u^i - u^{i-1} \|_{L^4_{tx}} < \frac{1}{2} \| u^{i-1} - u^{i-2} \|_{L^4_{tx}}$ (this is a contraction)

By induction if $\| u^1 \|_{L^4_{tx}} < \frac{1}{2} \left(\frac{1}{2\sqrt{2}} \right)$ (3)

then $\| u^i \|_{L^4_{tx}} \leq \left(1 - \frac{1}{2^i}\right) \left(\frac{1}{2\sqrt{2}}\right)$

$u^1 = S(t,0)\, u_0$ so $\| u^1 \|_{L^4_t(\mathbb{R}, L^4_x)} \lesssim \| \vec{u}_0 \|_{H^s \times H^{s-1}}$. so (3) holds for small energy T.

~~For two solutions u, v with data $\vec{u}_0, \vec{V}_0$~~

$$\cancel{\|u-v\|_{L^4_{tx}} \lesssim \left(\|u\|^2_{L^4_{tx}} + \|v\|^2_{L^4_{tx}}\right)\|u-v\|_{L^4_{tx}} + \|\vec{u}_0 - \vec{V}_0\|_{H^s \times H^{s-1}}}$$

$$\cancel{\lesssim \|\vec{u}_0 - \vec{V}_0\|_{H^s \times H^{s+1}} + \|u\|^2_{L^4_{tx}}\|u-v\|_{L^4_{tx}} + \|u-v\|^3_{L^4_{tx}}}.$$

~~Similarly,~~

For two solutions u, v with data $\vec{u}_0, \vec{V}_0$

$$\|\vec{u} - \vec{V}\|_{H^s \times H^{s-1}} \leq \|\vec{u} - S(t,0)\vec{u}_0\| + \|S(t,0)\vec{u}_0 - S(t,0)\vec{V}_0\| + \|S(t,0)\vec{V}_0 - \vec{V}\|$$

$\therefore \quad \vec{u}_0 \mapsto \vec{u}$ is continuous and $u_i \to u \in L^\infty(H^s \times H^{s-1})$.

For any $t<T$ ~~for~~ the same argument holds.

$$\| \vec{u} - S(t,0)\vec{u}_0 \|_{H^s \times H^{s-1}}(t) \lesssim \| N(u) \|_{L_t^{q'}([0,t], L_x^{r'})}$$
$$\lesssim \underbrace{\| u \|^3_{L_t^4([0,t], L_x^4)}}$$

This is a convergent integral.

$\therefore$ It is continuous in t.

$\therefore$ $\vec{u} : [0,T] \to H^s \times H^{s-1}$ is continuous at 0.

By time translation symmetry, $\vec{u} \in C([0,T] H^s \times H^{s-1})$. $\square$

For 2 solution u, v w/ data $\vec{u}_0, \vec{v}_0$
$\|\vec{u} - \vec{v}\| \le \|\vec{u} - S u_0\| + \|S u_0 - S v_0\| + \|S v_0 - v\|$ $\therefore$ $\vec{u}_0 \mapsto \vec{u}$ is cont.

Corollary (1) is GWP for small initial data.

pf ~~To run the~~ ~~We need (3)~~

~~We need (3):~~

~~If $\|S(t,0)u_0\|_{L_t^4([0,T], L_x^4)} < \frac{1}{2} \frac{1}{2\sqrt{c}}$.~~

~~$\|S(t,0)u_0\|_{L_t^4([0,\infty), L_x^4)}$~~ If $\|\vec{u}_0\|_{H^s \times H^{s-1}} \lesssim \frac{1}{2} \frac{1}{2\sqrt{c}}$

then by Strichartz $\|S(t,0)u_0\|_{L_t^4([0,\infty), L_x^4)} \le \frac{1}{2} \frac{1}{2\sqrt{c}}$

so (3) holds with $T = \infty$. $\square$

Corollary For $s > \frac{1}{2}$, $\Box u = |u|^2 u$ is WP with norm-dependence

pf Replace ~~$\|$ by~~ $\|u\|_{L^4_{tx}}$ by

$$\| \langle \nabla \rangle^{s-1/2} u \|_{L^4_{tx}} + \| u \|_{L_t^q L_x^4}$$

where $(4,q)$ is s-admissible.

By fractional Leibniz, the key estimate is

$$\| N(u^{i-1}) + N_u^{(i-2)} \|_{L^{4/3}} \lesssim \left(\|u\|_{L^4} + \|u\|_{L^4} \right) \| \langle \nabla \rangle^{s-1/2} (u^{i-1} - u^{i-2}) \|_{L^4_{tx}}.$$

~~A similar contraction argument shows that~~

$\|u^{i-1} - u^{i-2}\|_Y$

apply Hölder in time
since $q > 4$

$$\approx T^{\alpha} \left(\|u^{i-1}\|^{\beta}_{L^q_t L^4_x} + \|u^{i-2}\|^{\beta}_{L^q_t L^4_x} \right)$$

The contraction will run if

$$T^{\alpha} \quad \|S(t,0)u_0\|^{\beta}_{L^q_t L^4_x} \lesssim \frac{1}{2}$$

which holds by Strichartz if

$$T^{\alpha} \lesssim \|u_0\|_{H^s \times H^{s-1}} .$$

$\square$

Corollary

(a) for $-\partial_t^2 u = -\Delta u - |u|^2 u$ there are blow-ups

(b) $-\partial_t^2 u = -\Delta u + |u|^2 u$ GWP in H^1 if $s_c < 1$.

pf: (a) ODE trick

(b) The energy $\int \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{p+1}|u|^{p+1}\, dx > \|u\|_{H^1 \times H^0}$

is conserved.