OK

Thank organisers.

Maxwell field decay outside a Schwarzschild ~~black~~ hole

Relativity

(M^{1+3}, g)

M spacetime

g Lorentz pseudometric $\qquad$ $\overset{+---}{(-+++)}$ signature $\quad$ (like wave equ)

$g(V,V) < 0$ $\qquad$ V is timelike

$= 0$ $\qquad$ null $\qquad$ Say.

> 0 $\qquad$ spacelike

vacuum-Einstein equation

$$\boxed{Ric[g] = 0}$$

gravity is curvature (Riem & Ric)
matter produces curvature (Ric)
grav force is geodesic deviation

Lorentz $g \Rightarrow$ nonlinear hyperbolic PDE. $\quad$ start & evolve

Ex

- $\left(\mathbb{R}^{1+3}, \; -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right)$ Minkowski
  Laplace equ → wave equ.

- Schwarzschild:
  $\leftarrow -(1-2M/r)dt^2 + (1-2M/r)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$
  "black hole" (non-trivial, asymptotically flat, no ergosphere, extend through $r=2M$, tlike curve can enter but not leave
  [$\frac{\partial}{\partial t}$ & spherical symmetry just say]
  $r > 2M$ exterior $\qquad$ (block I)
  $r = 3M$ orbiting geodesics : trapping
  Kerr-Newman with $a=0$, $Q=0$, $M>0$

Conjecture: Black hole stability

Any perturbation of initial data of a Kerr-Newman black hole will evolve with a limit approaching one of the known members of the 3-parameter Kerr-Newman family.

(open dynamics: (M, a, Q) and external gravitational & EM radiation)

Maxwell equations

electromagnetic field

$F \in \Lambda^2(M)$ $\qquad\qquad$ $F_{\alpha\beta} = F_{[\alpha\beta]}$

$dF = 0$ $\qquad\qquad$ $\nabla_{[\gamma} F_{\alpha\beta]} = 0$

$d^*F = 0$ $\qquad$ Hodge uses metric $\qquad$ $\nabla^\alpha F_{\alpha\beta} = 0$

If $\hat{T}$ timelike unit vector

$\hat{X}, \hat{Y}, \hat{Z}$ oriented spacelike orthonormal

$E_{\hat{X}} = F(\hat{T}, \hat{X})$

$B_{\hat{X}} = F(\hat{Y}, \hat{Z})$

$= {}^*F(\hat{T}, \hat{X})$ $\qquad$ (Better to use spinors)

$$T_{\alpha\beta} = F_{\alpha\gamma} F_\beta{}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}$$

regular properties

$T_\alpha{}^\alpha = 0$ $\qquad$ traceless

Sets of Vectors

No way to measure F, need to choose, also for sym

Symmetries $\quad T = \partial/\partial t$

Θ_i rotation about axes.

$\mathbb{T} = \{T, \Theta_i\}$ $\qquad$ gather in sets

$\mathbb{O} = \{\Theta_i\}$

Basis $\quad \hat{T} = (1-2M/r)^{-1/2} \partial/\partial t, \ \hat{R} = (1-2M/r)^{1/2} \partial/\partial r, \ \hat{\Theta} = r^{-1} \partial/\partial\theta, \ \hat{\Phi} = \frac{1}{r\sin\theta} \partial/\partial\phi$

construct

Alternate radial coordinate

$$\frac{dr_*}{dr} = (1-2M/r) \in \mathbb{R} \quad \text{in exterior}$$

$r_* = 0 \iff r = 3M.$ $\qquad$ (measures radially has unit velocity along null geodesics)

- Formal wave-like equations & decay [Price, etc] Regge-Wheeler, Zerilli Teukolsky
- $\mathbb{R}^{1+3}$ stability [Christodoulou-Klainerman] (Major milestone, 1993, first real result in MR)
- t^{-1} pointwise wave decay on Schwarzschild [Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski]
  (roughly same time, technical result w error, St first correct proof, better at $r=3M$) Tataru

Now Maxwell, good step as a system like E eqn. cf CK

Thm There is a norm depending on the Maxwell field at its first 8 derivatives at $t=0$ such that for $2M < r_1 < r_2$ $\exists C : \forall\, t, r, \theta, \phi$ with $r \in [r_1, r_2]$

Vector field methods (pointwise diff from scattering) $|E| + |B| \leq t^{-1} C_{[r_1, r_2]} \| F(0) \|$.

Similar estimates at event horizon & null infinity.

energy-momentum tensor for some field ϕ $\qquad T_{\alpha\beta}$ $\qquad$ PDE from Lagrangian or var form follows

satisfies: $\qquad$ symmetric, $\nabla^\alpha T_{\alpha\beta} = 0 = \sum_{X \text{ in ONB}} (\nabla_X T)(X, \cdot)$

dominant energy condition $\forall X, Y$ time-like, future oriented $T(X,Y) \geq 0$

generalised momentum form associated to vector X $\qquad {}^{(X)}P = i_X T \qquad {}^{(X)}P_\alpha = T_{\alpha\beta} X^\beta$

generalised energy on hypersurface Σ^3 $\qquad E_X[\phi](\Sigma) = \int_\Sigma {}^{(X)}P \, dy$

$$= \int_\Sigma T_{\alpha\beta} X^\beta \, dy^\alpha$$

For a time foliation $\qquad E_X[\phi](t) = \int_{\{t = \text{const}\}} {}^{(X)}P \, dy$

$\because$ X time-like $\Rightarrow$ $E_X[\phi](t) \geq 0$ (and integrand non-negative)

A Killing vector is one which generates a symmetry / isometry

$$\mathcal{L}_X g = \text{sym}(\nabla X) = 0$$

(conformal Killing generates conformal sym $\mathcal{L}_X g = \text{sym}(\nabla X) = f(x)\, g$)

(i) By Stokes' theorem

$$X \text{ Killing} \Rightarrow E_X[\phi](t_2) - E_X[\phi](t_1) = \int_{t \in [t_1, t_2]} T^{\alpha\beta} \nabla_\alpha X_\beta \, d\text{vol} = 0$$

(Note T sym, IBP problems ignored)

E_X is conserved (say conserved quantity)

(ii) If ϕ satisfies a geometric PDE,

Y Killing $\Rightarrow$ $\mathcal{L}_Y \phi$ satisfies same PDE (operator commutes)

Consider $E_X[\mathcal{L}_Y^n \phi]$

~~Main achievement CK.~~

Spherical solutions

$$E_{\hat{R}} = \frac{q_E}{r^2}$$

$$B_{\hat{R}} = \frac{q_B}{r^2}$$

others are zero

$l = 0$ no dynamics

$l \geq 1$ decoupled

$$E_T[F](t) = \int_{\{t\}\times\mathbb{R}\times S^2} \frac{1}{2}\left(|E|^2 + |B|^2\right) \quad (1-2M/r)r^2\, dr_*\, d^2\omega$$

$$T \text{ Killing} \Rightarrow E_T[F](t_*) = E_T[F](0)$$

L^2_{loc} boundedness

$$K = (t^2 + r_*^2)\partial_t + 2tr_*\partial_{r_*}$$

analogue of conformal Killing field from $\mathbb{R}^{1+3}$

$$E_K[F](t) = \int\left[\underbrace{(t^2+r_*^2)}_{\text{grows like } t^2}\left(|E|^2+|B|^2\right) + 2tr_*\left(E_{\hat\theta}B_{\hat\phi} - E_{\hat\phi}B_{\hat\theta}\right)\right] (1-2M/r)r^2\, dr_*\, d^2\omega$$

grows like t^2. In fixed regions only first part matters.

$$\frac{d}{dt}E_K[F] \underset{}{=} \int 4\cancel{2}t\left(1 - \frac{r_*}{r}(1-3M/r)\right)\left(E_{\hat R}^2 + B_{\hat R}^2\right) (1-2M/r)r^2\, dr_*\, d^2\omega$$

$= \int T\ \mathrm{sym}\nabla K$

reveals trapping

>0 in compact region.

only 2 of 6 components

$r^2E_{\hat R}$, $r^2B_{\hat R}$ each satisfy a wave equation

$$\Rightarrow E_K[F](t) \le C\left(E_K[F](0) + \sum_{n=0}^{9}[\mathcal{L}^n_\Omega F](0)\right)$$

t^{-1} decay in L^2_{loc}

Idea for wave analysis

Find $g\partial_{r_*}$ with

- g bounded
- $g\partial_{r_*}$ has a positive commutator

$$\Rightarrow \int_{\text{exterior}} \frac{1}{\langle r_* \rangle^4} |u|^2 \, d^4\mathrm{vol} \leq E_{g\partial_*}\Big|_{-\infty}^{\infty} \leq C E_T .$$

r_α !

$$K = (t^2 + r_*^2)\partial_t + 2t r_* \partial_{r_*}$$

$$\left[(t^2+r_*^2)(|E|^2+|B|^2) + 2t r_*(E_{\hat\theta}B_{\hat\phi} - E_{\hat\phi}B_{\hat\theta})\right]$$

$$E_K[F] = \int \left[(t+r_*)^2|\phi_1|^2 + (t^2+r_*^2)|\phi_0|^2 + (t-r_*)^2|\phi_{-1}|^2\right](1-2M/r) r^2 dr\, d^2\omega$$

for fixed r, large t like $t^2 \times$ Energy

$$\frac{d}{dt}E_K[F] = \int 2t\left(1 - \frac{r_*}{r}(1-3M/r)\right)|\phi_0|^2 \,(1-2M/r) r^2 dr\, d^2\omega$$

$$(E_{\hat r}^2 + B_{\hat r}^2)$$

$r^2\phi_0$ satisfies a wave equation

$$\Rightarrow E_K[F](t) \le C\left(E_K[F](0) + \sum_{k=0}^{n} E_K[\mathcal{L}^k_{\Omega} F](0)\right)$$

t^{-1} decay in L^2_{loc}

Pointwise decay

$E_K[\mathcal{L}_T F]$ is bounded   Want to apply Sobolev

control all derivatives except radial.

$$|F|_* = \sum_{X_1, X_2 \in \{T, R, \Theta, \Phi\}} |F(X,Y)|$$

$$\partial_{r_*}(F(X,Y)) = (\nabla_R F)(X,Y) + F(\nabla_R X, Y) + F(X, \nabla_R Y)$$

Maxwell!

$$= -\nabla_X F(Y,R) - \nabla_Y F(R,X) + O(|F|_*)$$

$$= -X(F(Y,R)) - Y(F(R,X)) + O(|F|_*)$$

$$= -(\mathcal{L}_X F)(Y,R) - (\mathcal{L}_Y F)(R,X) + O(|F|_*)$$

$$= O(|\mathcal{L}_T F|_*) + O(|F|_*)$$

similarly for other components

For $2M < r_1 < r < r_2$

$$|F(X,Y)|^2 < C\, \|F(X,Y)\|^2_{H^3_{loc}}$$ Sobolev

$$< C\int \sum_{n=0}^{3} |\mathcal{L}^n_T F|^2_* \, d^3x$$ derivative trading

$$< C\, t^{-2} \sum_{k=0}^{3} E_K[\mathcal{L}^n_T F](t)$$

$$|F(X,Y)| \lesssim t^{-1}$$