Decay for the wave equation outside a slowly rotating Kerr black hole

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Hidden symmetries and Kerr wave decay

Joint with Lars Andersson.

- Kerr spacetime
  - parameters: M mass and Ma angular momentum.
  - rotating black hole, expected end state.
  - black hole for $|a| \leq M$; $a = 0$ is Schwarzschild.
- Wave: $\nabla^\alpha \nabla_\alpha \psi = 0$, decoupled, important equation, model.
- Goal: robust tools (hopefully) for Kerr stability.
- We consider $|a| \ll M$, exterior $r > r_+$.
- Result: $t^{-1+|a|C}$ decay for $|a| \ll M$ [arXiv:0908.2265].

Kerr wave decay: other results

- Minkowski stability: Christodoulou-Klainerman, Lindblad-Rodnianski
- $|a| \leq M$, mode decay: Finster-Kamran-Smoller-Yau
- $a = 0$, integrated decay/ local energy decay/ Morawetz estimate and decay rate estimates: Łaba- Soffer, B- Soffer (but gap), B- Sterbenz, Dafermos- Rodnianski, Metcalfe- Marzuola- Tataru- Tohaneanu, Luk.
  Donninger- Schlag- Soffer.
- $a \ll M$: Boundedness, **integrated decay/ local energy estimate/ Morawetz estimate**, decay rates.
  Dafermos-Rodnianski, Tataru-Tohaneanu, Tataru (t^{-3})

Integrated decay/ local energy estimates/ Morawetz

For some localising weight $\mathbf{1}_A$ independent of t,

$$\int_0^\infty \int_{\text{space}} \mathbf{1}_A |\partial\psi|^2 \mathrm{d}^3x \mathrm{d}t \le C \int_{\text{space}} |\partial\psi|^2 \mathrm{d}^3x.$$

General relativity in 1 slide

- $\mathcal{M}$: space-time manifold.
- g: Lorentz (-,+,+,+ signature) pseudometric.
  - Time-like vector: $g(v, v) < 0$,
  - null vector: $g(v, v) = 0$,
  - space-like vector: $g(v, v) > 0$.
  - Summation convention $g(v, v) = g_{\alpha\beta} v^\alpha v^\beta$.
  - Curve length: $\int \sqrt{|g(\dot\gamma, \dot\gamma)|} ds$.
- (Vacuum) Einstein equation:

$$\mathrm{Ric}[g] = 0.$$

- Globally hyperbolic: $\mathcal{M} = \mathbb{R} \times \Sigma$ (topologically).

Energy-momentum tensor

Energy-momentum tensor:

$$T[\psi]_{\alpha\beta} = \nabla_\alpha\psi\nabla_\beta\psi - g_{\alpha\beta}(\nabla_\gamma\psi\nabla^\gamma\psi).$$

Given a vector-field X, 4-momentum

$$\begin{aligned} {}^{(X)}P[\psi]_\alpha &= T[\psi]_{\alpha\beta}X^\beta, \\ E_X[\psi](\Sigma) &= \int_\Sigma {}^{(X)}P[\psi]_\alpha \mathrm{d}\nu^\alpha. \end{aligned}$$

Assume spacetime is foliated by Σ_t with timelike, future-oriented normal

$$E_X[\psi](\Sigma_t) = E_X[\psi](t) = E_X[\psi] = E_X(t)$$

Energy-momentum properties

Properties:

1. If $\mathbf{T}$ timelike,

   then $E_{\mathbf{T}} \geq 0$.

2. We call S a (generalised) symmetry when
   $\nabla^\alpha \nabla_\alpha \psi = 0 \implies \nabla^\alpha \nabla_\alpha S\psi = 0$,
   If S is a symmetry,

   then $E_X[S\psi]$ has the same properties as $E_X[\psi]$.

3. If X is Killing,

   then $E_X(t_2) = E_X(t_1)$.

   Otherwise: $E_X(t_2) - E_X(t_1) = \int T[\psi]_{\alpha\beta} \nabla^{(\alpha} X^{\beta)} \mathrm{d}^4 \mu_g$.

Minkowski example

∂_t is timelike and Killing. It generates a positive conserve energy:

$$E_{\partial_t}[\psi] = \int_{\mathbb{R}^3} |\partial_t\psi|^2 + |\nabla\psi|^2 \mathrm{d}^3 x \geq \|\psi\|^2_{\dot{H}^1}.$$

Translations are symmetries:

$$E_{\partial_t}[\partial_x\psi] + E_{\partial_t}[\partial_y\psi] + E_{\partial_t}[\partial_z\psi] \geq \|\psi\|^2_{\dot{H}_2}.$$

$$\ldots$$

Sobolev.
Rotations are symmetries.

Extra Lagrangian terms

$$
{}^{(q)}P_\alpha = q\psi\nabla_\alpha\psi - \frac{1}{2}(\nabla_\alpha q)|\psi|^2,
$$

$$
\nabla^{\alpha(q)}P_\alpha = q\nabla^\alpha\psi\nabla_\alpha\psi - \frac{1}{2}(\nabla^\alpha\nabla_\alpha q)|\psi|^2.
$$

$$
E_q = {}^{(X)}P[\psi]_\alpha \mathrm{d}\nu^\alpha,
$$

$$
E_q(t_2) - E_q(t_1) = \int \nabla^{\alpha(q)}P_\alpha \mathrm{d}^4\mu_g.
$$

Geometry of Schwarzschild and Kerr

- Mass M, rotational parameter a.
- Schwarzschild is $a = 0$, subcritical Kerr is $|a| < M$.
- Spherical co-ordinates, (t, r, θ, ϕ):

$$\begin{aligned} g = & -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 \\ & + \Sigma d\theta^2 + \left((r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right) \frac{\sin^2\theta}{\Sigma} d\phi^2, \\ \Sigma = & r^2 + a^2\cos^2\theta, \\ \Delta = & r^2 - 2Mr + a^2. \end{aligned}$$

- Exterior: $r > r_+ = M + \sqrt{M^2 - a^2}$.
- Symmetries: ∂_t, ∂_ϕ.

Trapping

Null geodesics γ:
Conserved quantities: $E = g(\dot\gamma, \partial_t)$, $L_z = g(\dot\gamma, \partial_\phi)$, $0 = g(\dot\gamma, \dot\gamma)$,
and Carter constant:

$$\mathcal{Q} = g(\dot\gamma, \partial_\theta)^2 + \cot^2\theta L_z^2 + a^2 \sin^2\theta E^2.$$

Complete separation:

$$\Sigma^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \mathcal{R}(r; M, a; E, L_z, Q),$$

$$\mathcal{R}(r; M, a; E, L_z, Q) = -(r^2 + a^2)^2 E^2 - (4aMr)EL_z + \Delta\mathcal{Q} + (\Delta - a^2)L_z^2. \quad (1)$$

ODE analysis, turning points, orbits when $\mathcal{R} = \partial_r\mathcal{R} = 0$,
hyperbolic orbits $\partial_r^2\mathcal{R} < 0$, at $r = 3M + O(a)$. c.f. (2)

Kerr Problems

Problems:

1. No timelike, Killing vector: no positive, conserved energy.
2. $\partial_t, \partial_\phi$ only Killing vectors:
   $E_{\mathbf{T}}[k^n u]$ doesn't control Sobolev norms,
3. Photon orbits: "trapping".
   Can't prove Morawetz/ local energy estimate using a vector field.
4. No ∂_t to build $\mathbf{K}$.
5. Lack of pleasant null hypersurfaces.

Energy

Use blended energy

- Stationary vector field timelike for r large ∂_t.
- Null generator extension timelike for r near r_+ $\partial + \omega_H \partial_\phi$.
- For $|a|$ small, overlap.

Let

$$T_\chi = \partial_t + \chi \omega_H \partial_\phi.$$

Timelike in full exterior.
Failure to be conserved controlled by Morawetz (local decay) estimate.

The wave equation

The wave equation:

$$\left(\partial_r \Delta \partial_r + \frac{1}{\Delta}\mathcal{R}(r; M, a; \partial_t, \partial_\phi, \mathcal{Q})\right)\psi = 0 \qquad (2)$$

where $\Delta = r^2 - 2Mr + a^2$ and

$$\mathcal{Q} = \frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta + \cot^2\theta \partial_\phi^2 + a^2 \sin^2\theta \partial_t^2.$$

c.f. (1).

Symmetry algebra and higher energies

The set of symmetries forms a graded algebra.

e.g. Iterated Lie derivatives along Killing vectors are symmetries.

Let $\mathbb{S}_n$ be a basis of grade n and

$$E_{X,n+1}[\psi] = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}_i} E_X[S\psi],$$

$$|\psi|_n^2 = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}_i} |S\psi|^2.$$

Symmetry algebra for Kerr spacetime

In Kerr, there are Killing vectors ∂_t and ∂_ϕ.
There is also the (hidden) symmetry:

$$\mathcal{Q}$$

Order n generators:

$$\begin{aligned}\mathbb{S}_n =&\{\partial_t^{n_t}\partial_\phi^{n_\phi}\mathcal{Q}^{n_Q} \mid n_t, n_\phi, n_Q \in \mathbb{N}, n_t + n_\phi + 2n_Q = n\},\\ \mathbb{S}_0 =&\{\mathrm{Id}\},\ \ \mathbb{S}_1 = \{\partial_t, \partial_\phi\},\ \ \mathbb{S}_2 = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, \mathcal{Q}\} = \{S_{\underline{a}}\}.\end{aligned}$$

Thus,

$$|\Delta_{S^2}\psi| \leq |\mathcal{Q}\psi| + |\partial_\phi^2\psi| + |\partial_t^2\psi| \leq |\psi|_2,$$

and for $r > r_0 > r_+$,

$$|\psi(t, r, \theta, \phi)| \lesssim E_{\mathbf{T},3}(t).$$

Morawetz (local energy) estimate idea

$$A = \mathcal{F}\partial_r$$

(additional terms)
Illustrate method by integration by parts:

$$\begin{aligned}
0 =& (\mathcal{F}\partial_r\psi)(\partial_r^2\psi + \mathcal{R}\psi) \\
=& (\partial_r\psi)\frac{1}{2}(\mathcal{F}')(\partial_r\psi) + \psi(-\mathcal{F})\left(\partial_r\mathcal{R}\right)\psi \\
&+ \text{l.o.t.s} \\
&+ \partial_t(\mathcal{F}\psi'\partial_t\psi) + \partial_r((1-2M/r)(\text{terms})).
\end{aligned}$$

$\Delta\mathcal{F}$ bounded $\implies \int_{\Sigma_t} |\Delta\mathcal{F}(\partial_r\psi)\partial_t\psi| \mathrm{d}^3\mu \leq E_{\mathbf{T}}$.

Higher energies and momenta for $\mathbb{S}_2$ vectors

Let

$$\begin{aligned} T[\psi_1,\psi_2]_{\alpha\beta} =& (1/4)\left(T[\psi_1+\psi_2]_{\alpha\beta} - T[\psi_1-\psi_2]_{\alpha\beta}\right), \\ T[\psi_1]_{\underline{ab}\alpha\beta} =& T[S_{\underline{a}}\psi, S_{\underline{b}}\psi]_{\alpha\beta}. \end{aligned}$$

Given $X^{\underline{ab}}$, let

$$\begin{aligned} {}^{(X^{\underline{ab}})}P[\psi]_\alpha =& T[\psi]_{\underline{ab}\alpha\beta} X^{\underline{ab}\beta}, \\ E_{X^{\underline{ab}}}[\psi] =& \int {}^{(X^{\underline{ab}})}P[\psi]_\alpha \mathrm{d}\nu^\alpha. \end{aligned}$$

Morawetz estimate

Wave equation takes form $\left(\partial_r \Delta \partial_r - \Delta^{-1}\mathcal{R}(r)^{\underline{a}} S_{\underline{a}}\right)\psi = 0$.
$(\Delta = r^2 - 2Mr + a^2)$

Build $A^{\underline{ab}}$ from $\mathcal{R}^{\underline{a}}$, $\mathcal{L}^{\underline{b}}$, and weights z and w, where

$$A^{\underline{ab}} = zw\partial_r(z\Delta^{-1}\mathcal{R}^{(\underline{a}})\mathcal{L}^{\underline{b})},$$
$$\mathcal{L}^{\underline{a}} S_{\underline{a}} = \partial_t^2 + \partial_\phi^2 + \mathcal{Q}.$$

Suitable choices $\implies E_{T_\chi,3} \geq C E_{A^{\underline{ab}}}$.
(Lower-order terms use $q^{\underline{ab}} = (1/2)(\partial_r z) w \partial_r (z\Delta^{-1}\mathcal{R}^{(\underline{a}})\mathcal{L}^{\underline{b})})$

Morawetz estimate (cont.)

Get $T_{\underline{ab}\alpha\beta}\nabla^\alpha A^{\underline{ab}\beta}$ like

$$\Delta^{3/2}z^{1/2}\left(\partial_r w\frac{z^{1/2}}{\Delta^{1/2}}\left(-\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\right)(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)$$
$$+w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)$$
$$+\frac{1}{4}\left(\partial_r\Delta\partial_r z\left(\partial_r w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\right)\right)\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).$$

So

$$E_{T_\chi,3}(t_2)+E_{T_\chi,3}(t_1)$$
$$\geq C\int\frac{1}{r^2}|\partial_r\psi|_2^2+\mathbf{1}_{r\not\approx 3M}\frac{1}{r^3}\left(|\partial_t\psi|_2^2+|\not\nabla\psi|_2^2\right)+\frac{1}{r^4}|\psi|_2\mathrm{d}^4\mu_g.$$

The ODE argument

The sum of the first and third terms must be positive. Ignoring $S_{\underline{a}}$ and taking $a = 0$, these are

$$\int_{r_+}^{\infty} \frac{\Delta^2}{r^2(r^2+a^2)}(\partial_r\psi)^2 + \frac{1}{6}\frac{9r^2 - 46Mr + 54M^2}{r^4}|\psi|^2 \mathrm{d}r. \qquad (3)$$

Variational argument, existence of positive solution to associated ODE gives lower bound by

$$\epsilon \int_{r_+}^{\infty} \frac{\Delta^2}{r^2(r^2+a^2)}(\partial_r\phi)^2 + \frac{1}{r^2}|\psi|^2 \mathrm{d}r.$$

(c.f. 28)

Bounded energy argument and local energy decay

$$\begin{aligned} E_{T_\chi,3}(t_2) - E_{T_\chi,3}(t_1) \geq & |a| C \int ((localisation)) |\partial^3\psi|^2 \mathrm{d}^4\mu_g \\ \geq & |a| C (E_{T_\chi,3}(t_2) + E_{T_\chi,3}(t_1)). \end{aligned}$$

$$E_{T_\chi,3}(t_2) \leq \frac{1+|a|C}{1-|a|C} E_{T_\chi,3}(t_1)$$

$$E_{T_\chi,3}(t_1) \geq$$
$$C \int \frac{1}{r^2} |\partial_r \psi|_2^2 + \mathbf{1}_{r \not\approx 3M} \frac{1}{r^3} \left(|\partial_t \psi|_2^2 + |\not\nabla \psi|_2^2 \right) + \frac{1}{r^4} |\psi|_2 \mathrm{d}^4\mu_g$$

Decay estimate

Use

$$\mathbf{K} = (t^2 + r_*^2) T_\chi + 2 \left(\frac{(r^2+a^2)^2 - a^2 \sin^2\theta \Delta}{(r^2+a^2)^2} \right) t r_* \partial_{r_*}.$$

$$\frac{\mathrm{d} r_*}{\mathrm{d} r} = \frac{\Delta}{r^2+a^2}.$$

- Almost a conformal symmetry
- Most of $T_{\alpha\beta}\nabla^\alpha \mathbf{K}^\beta$ can be controlled by Morawetz and bootstrap.
- Remains $at^2(\partial_{r_*}\psi)(\partial_\phi\psi)$ in some region. Get $\frac{\mathrm{d}}{\mathrm{d}t} E_\mathbf{K} \leq a E_\mathbf{K} + \dots$. Integrating factor gives $E_\mathbf{K}(t) \leq C_1 t^{C_2|a|} E_\mathbf{K}(0) + \dots$.
- For $r \in [r_1, r_2] \subset (r_+, \infty)$: $|\psi(t,r,\theta,\phi) \lesssim t^{-1+C|a|}$(init data).

Decay near the event horizon

Use

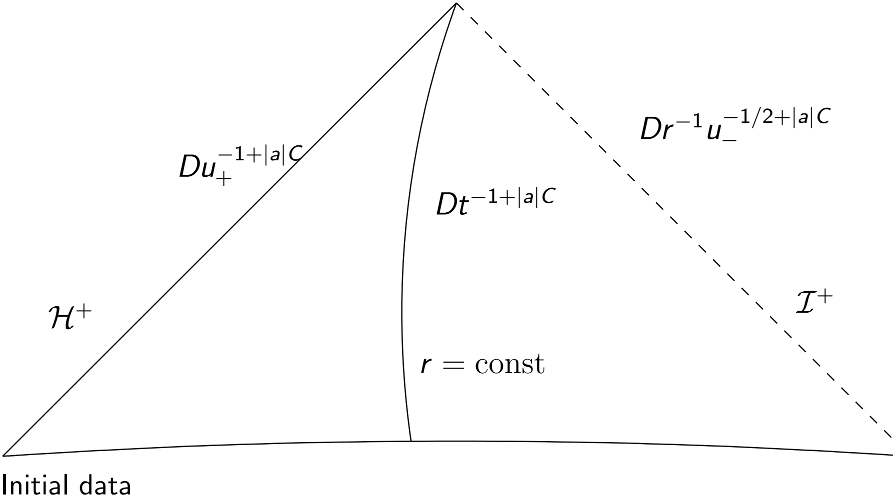
$$(\partial_- \psi)\mathrm{d}u_-,$$

and apply Stokes' theorem in a region bounded by $t = 0$, $u_+ = \text{const}$, and u_- between two constants.
On the boundary get something from initial data, nothing on two sides, and the difference between ψ at the desired point at $r = 3M$. The contribution from Stokes' theorem is controlled by Morawetz and earlier decay estimates.

Decay near null infinity

Use hyperboloidal hypersurfaces and the **K** energy to approach null infinity.
The hyperboloidal hypersurfaces have normal which is timelike, but degenerates at a rate of r^{-2} with respect to ∂_t and ∂_{r_*}.

Theorem: On Kerr with $|a| \ll M$, $u_\pm = t \pm r_*$



$$D^2 = \|\psi\|(0)^2 = E_{T_\chi,9}(0) + E_{\mathbf{K},5}(0) + E_{n,3}(0).$$

Maxwell equation

Maxwell equation:

$$F_{\alpha\beta} = F_{[\alpha\beta]}, \qquad \nabla^\alpha F_{\alpha\beta} = 0, \qquad \nabla_{[\alpha} F_{\beta\gamma]} = 0.$$

Spin-lowering

$$\begin{aligned} \phi_0 &\sim *F(\Theta,\Phi) + iF(\Theta,\Phi), \\ \Theta &= \partial_\theta, \\ \Phi &= \frac{1}{\sin\theta}\partial_\phi + a\sin\theta\partial_t, \\ \frac{1}{\sin\theta}\Theta\sin\theta\Theta + \frac{1}{\sin^2\theta}\Phi\Phi &= \mathcal{Q} + \partial_\phi^2 + 2\partial_\phi\partial_t. \end{aligned}$$

Ipser-Fackerel equation:

$$(\Box + V)((r - ia\cos\theta)^{-1}\phi_0) = 0.$$

Maxwell (cont.)

There are explicitly known stationary solutions.
In Schwarzschild, these are spherically symmetric.

The extra potential V requires a stronger Hardy estimate. c.f. (3).
In Schwarzschild, project out $l = 0$ stationary solution, and use $|\not\nabla\psi|^2 > |\psi|^2$ to get more positivity so the Hardy estimate is sufficient.