Decay for the wave equation outside a slowly rotating Kerr black hole

Pieter Blue

19 August 2010
Philadelphia: SIAM nonlinear waves and coherent structures

Joint with Lars Andersson.

- Kerr spacetime
  - parameters: M mass and Ma angular momentum.
  - rotating black hole, expected end state.
  - black hole for $|a| \leq M$; $a = 0$ is Schwarzschild.
- Wave: $\nabla^{\alpha}\nabla_{\alpha}\psi = 0$, decoupled, important equation, model.
- Goal: robust tools (hopefully) for Kerr stability.
- We consider $|a| \ll M$, exterior $r > r_{+}$.
- Result: $t^{-1+|a|C}$ decay for $|a| \ll M$ [arXiv:0908.2265].

Kerr wave decay: other results

- $|a| \leq M$, mode decay: Finster-Kamran-Smoller-Yau
- $a = 0$, integrated decay/ local energy decay/ Morawetz estimate and decay rate estimates: Łaba- Soffer, B- Soffer (but gap), B- Sterbenz, Dafermos- Rodnianski, Metcalfe- Marzuola- Tataru- Tohaneanu, Luk.
  Donninger- Schlag- Soffer.
- $a \ll M$: Boundedness, **integrated decay/ local energy estimate/ Morawetz estimate**, decay rates.
  Dafermos-Rodnianski, Tataru-Tohaneanu, Tataru (t^{-3})

Integrated decay/ local energy estimates/ Morawetz

For some localising weight $\mathbf{1}_A$ independent of t,

$$\int_0^\infty \int_{\text{space}} \mathbf{1}_A |\partial\psi|^2 \mathrm{d}^3x \mathrm{d}t \leq C \int_{\text{space}} |\partial\psi|^2 \mathrm{d}^3 x.$$

General relativity in 1 slide

- $\mathcal{M}$: space-time manifold.
- g: Lorentz (-,+,+,+ signature) pseudometric.
  - Time-like vector: $g(v,v) < 0$,
  - null vector: $g(v,v) = 0$,
  - space-like vector: $g(v,v) > 0$.
  - Summation convention $g(v,v) = g_{\alpha\beta}v^\alpha v^\beta$.
  - Curve length: $\int \sqrt{|g(\dot\gamma,\dot\gamma)|}ds$.
- (Vacuum) Einstein equation:

$$\mathrm{Ric}[g] = 0.$$

Energy-momentum tensor:

$$T[\psi]_{\alpha\beta} = \nabla_\alpha\psi\nabla_\beta\psi - g_{\alpha\beta}(\nabla_\gamma\psi\nabla^\gamma\psi).$$

Given a vector-field X, 4-momentum

$$\begin{aligned} {}^{(X)}P[\psi]_\alpha &= T[\psi]_{\alpha\beta}X^\beta, \\ E_X[\psi](\Sigma) &= \int_\Sigma {}^{(X)}P[\psi]_\alpha \mathrm{d}\nu^\alpha. \end{aligned}$$

Assume spacetime is foliated by Σ_t with timelike, future-oriented normal

$$E_X[\psi](\Sigma_t) = E_X[\psi](t) = E_X[\psi] = E_X(t)$$

Energy-momentum properties

Properties:

1. If $\mathbf{T}$ timelike,

   then $E_{\mathbf{T}} \geq 0$.

2. We call S a (generalised) symmetry when
   $\nabla^\alpha \nabla_\alpha \psi = 0 \implies \nabla^\alpha \nabla_\alpha S\psi = 0$,
   If S is a symmetry,

   then $E_X[S\psi]$ has the same properties as $E_X[\psi]$.

3. If X is generates a symmetry,

   then $E_X(t_2) = E_X(t_1)$.

   Otherwise: $E_X(t_2) - E_X(t_1) = \int T[\psi]_{\alpha\beta} \nabla^{(\alpha} X^{\beta)} \mathrm{d}^4\mu_g$.

Geometry of Schwarzschild and Kerr

- Spherical co-ordinates, (t, r, θ, ϕ):

$$
\begin{aligned}
g =& -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mra\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 \\
&+ \Sigma d\theta^2 + \left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right)\frac{\sin^2\theta}{\Sigma}d\phi^2, \\
\Sigma =& r^2 + a^2\cos^2\theta, \\
\Delta =& r^2 - 2Mr + a^2.
\end{aligned}
$$

- Exterior: $r > r_+ = M + \sqrt{M^2 - a^2}$.
- Symmetries: ∂_t, ∂_ϕ.

Kerr Problems

Problems:

1. No timelike, Killing vector: no positive, conserved energy.
2. $\partial_t, \partial_\phi$ only Killing vectors: $E_{\mathbf{T}}[k^n u]$ doesn't control Sobolev norms,
3. Photon orbits: "trapping" at more than one r. Can't prove Morawetz/ local energy estimate using a vector field.

Energy

Use blended energy

- Stationary vector field timelike for r large ∂_t.
- Null generator extension timelike for r near r_+ $\partial_t + \omega_H \partial_\phi$.
- For $|a|$ small overlap.

Let

$$T_\chi = \partial_t + \chi \omega_H \partial_\phi.$$

Timelike in full exterior.
Failure to be conserved controlled by Morawetz (local decay) estimate.

Hidden symmetries

Hidden symmetry from Carter Killing 2-tensor

$$\mathcal{Q} = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{\cos^2\theta}{\sin^2\theta}\partial_\phi^2 + a^2\sin^2\theta\partial_t^2.$$

Symmetry algebra

$$\mathbb{S}_2 = \{S_{\underline{a}}\}_{\underline{a}} = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, \mathcal{Q}\},$$

$$|\Delta_{S^2}u|^2 \leq |\mathcal{Q}u|^2 + |\partial_\phi^2 u|^2 + |\partial_t^2 u|^2,$$

$$E_{\mathbf{T}}[\Delta_{S^2}u] \leq \sum_{\underline{a}} E_{\mathbf{T}}[S_{\underline{a}}u].$$

Morawetz (local energy) estimate idea

Wave equation

$$0 = \left(\partial_r \Delta \partial_r + \frac{1}{\Delta}\mathcal{R}\right)\psi = \left(\partial_r \Delta \partial_r + \frac{1}{\Delta}\mathcal{R}(r)^{\underline{a}} S_{\underline{a}}\right)\psi.$$

Illustrate method by integration by parts (roughly):

$$\begin{aligned} A =& \mathcal{F}\partial_r \\ 0 =& (\mathcal{F}\partial_r\psi)(\partial_r^2\psi + \mathcal{R}\psi) \\ =& (\partial_r\psi)\frac{1}{2}(\mathcal{F}')(\partial_r\psi) + \psi(-\mathcal{F})\left(\partial_r\mathcal{R}\right)\psi \\ &+ \text{l.o.t.s} \\ &+ \partial_t(\mathcal{F}\psi'\partial_t\psi) + \partial_r(\Delta(\text{terms})). \end{aligned}$$

$\Delta\mathcal{F}$ bounded $\implies \int_{\Sigma_t} |\Delta\mathcal{F}(\partial_r\psi)\partial_t\psi| \mathrm{d}^3\mu \leq E_{\mathbf{T}}$.

Higher energies and momenta for $\mathbb{S}_2$ vectors

Let

$$E_{X,n+1}[\psi] = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}_i} E_X[S\psi],$$

$$|\psi|_n^2 = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}_i} |S\psi|^2.$$

Let

$$T[\psi_1,\psi_2]_{\alpha\beta} = (1/4)\left(T[\psi_1+\psi_2]_{\alpha\beta} - T[\psi_1-\psi_2]_{\alpha\beta}\right),$$
$$T[\psi_1]_{\underline{ab}\alpha\beta} = T[S_{\underline{a}}\psi, S_{\underline{b}}\psi]_{\alpha\beta}.$$

Given $\mathbb{S}_2$ vector $X^{\underline{ab}}$, let

$$^{(X^{\underline{ab}})}P[\psi]_\alpha = T[\psi_1]_{\underline{ab}\alpha\beta} X^{\underline{ab}\beta},$$

$$E_{X^{\underline{ab}}}[\psi] = \int {}^{(X^{\underline{ab}})}P[\psi]_\alpha \mathrm{d}\nu^\alpha.$$

Morawetz estimate

Morawetz $\mathbb{S}_2$ vector field

$$A^{\underline{ab}} = w\left(\partial_r \frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\mathcal{L}^{\underline{b}}\partial_r,$$
$$\mathcal{L} = \epsilon\partial_t^2 + \partial_\phi^2 + \mathcal{Q}.$$

Get $T_{\underline{ab}\alpha\beta}\nabla^\alpha A^{\underline{ab}\beta}$ (plus corrections) like

$$\Delta^{3/2}z^{1/2}\left(\partial_r w\frac{z^{1/2}}{\Delta^{1/2}}\left(-\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\right)(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)$$
$$+ w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)$$
$$+\frac{1}{4}\left(\partial_r\Delta\partial_r z\left(\partial_r w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\right)\right)\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).$$

Bounded energy argument and local energy decay

$$E_{T_\chi,3}(t_2)+E_{T_\chi,3}(t_1)$$
$$\geq C\int \frac{1}{r^2}|\partial_r\psi|_2^2 + \mathbf{1}_{r\not\sim 3M}\frac{1}{r^3}\left(|\partial_t\psi|_2^2+|\nabla\!\!\!/\psi|_2^2\right)+\frac{1}{r^4}|\psi|_2 \mathrm{d}^4\mu_g.$$

$$\begin{aligned} E_{T_\chi,3}(t_2)-E_{T_\chi,3}(t_1) \leq& |a|C\int(\text{localisation})|\partial^3\psi|^2\mathrm{d}^4\mu_g \\ \leq& |a|C(E_{T_\chi,3}(t_2)+E_{T_\chi,3}(t_1)). \end{aligned}$$

$$E_{T_\chi,3}(t_2)\leq\frac{1+|a|C}{1-|a|C}E_{T_\chi,3}(t_1)$$

Theorem

For $|a| < a_0$, if ψ satisfies the Kerr wave equation then $\exists C$:

$$E_{T_\chi,3}(t) + \int \frac{1}{r^2}|\partial_r\psi|_2^2 + \mathbf{1}_{r\not\approx 3M}\frac{1}{r^3}\left(|\partial_t\psi|_2^2 + |\not\nabla\psi|_2^2\right) + \frac{1}{r^4}|\psi|_2 \mathrm{d}^4\mu_g \leq CE_{T_\chi,3}(0).$$

Furthermore, $\exists c$ such that for $r_+ < r_1 < r_2 < \infty \exists C : \forall t \in \mathbb{R}, r \in (r_1, r_2), (\theta, \phi) \in S^2$

$$|\psi(t, r, \theta, \phi)| \leq Ct^{-1+C|a|} \left(E_{T_\chi,9}(0) + E_{\mathbf{K},7}(0) + E_{n,3}(0)\right).$$

Similarly for $r \to r_+$ and $r \to \infty$.

Decay rate estimate

- Use vector field $\mathbf{K} = (t^2 + r_*^2)\partial_t + \left(\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{(r^2+a^2)^2}\right) 2tr_*\partial_{r_*}$ to cancel top-order terms.
- Still a high-order term with coefficient a to be controlled: lose $C|a|$ in decay rate.
- Use hyperbolic, instead of null, surfaces to get decay near null infinity.
- Use Sterbenz form $(\partial_+\psi)\mathrm{d}u_+$, Stokes' theorem, and Morawetz to extend decay to event horizon.