Hidden symmetries and decay for the wave equation outside a Kerr black hole

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Overview

- Joint with Lars Andersson
- Kerr: rotating black hole, expected end state
- Wave: $\nabla^\alpha \nabla_\alpha u = 0$, important equation, model
- Problems:
  - No positive, conserved energy
  - Complicated trapping
  - Not enough symmetries to generate useful higher energies

History I

Many including

- Price ($a = 0$, decay rate prediction)
- Wald, Kay-Wald ($a = 0$, boundedness)
- Kay-Dimock ($a = 0$, scattering)
- Whiting (no exponential growth)
- Bachelot, Nicolas, Häfner (scattering, Morawetz-like propogation observable)
- Finster-Kamran-Smoller-Yau (representation, decay, energy extraction)

Morawetz history

- Łaba-Soffer (Schrödinger decay)
- B.-Soffer (wave)
- B.-Sterbenz (first correct proof)
- Dafermos-Rodnianski (independent correction, Y)
- Marzoula-Metcalfe-Tataru-Tohaneanu (Strichartz, interior estimate)
- Dafermos-Rodnianski ($a \neq 0$, energy bound)
- Tataru-Tohaneanu (independent proof, $a \neq 0$, energy bound)
- Dafermos-Rodnianski ($a \neq 0$, decay)

Conventions

- Signature $(-,+,+,+)$
- Take (M^{1+3}, g) foliated by surfaces indexed by t.
- $\not\nabla$ represents angular derivatives.
- $\langle r \rangle = (1 + r^2)^{1/2}$.

Vector-field method I: Momentum generating vector-fields

- Particle case: For a null geodesic γ, consider $\dot\gamma_\mu$ and:
  - Given X^α,

$$e_X[\gamma](t) = -X^\alpha\dot\gamma_\alpha(t)$$

  - If X is (future-oriented) time-like, then $e_X \geq 0$.
  - If X is Killing ($\nabla^{(\alpha}X^{\beta)} = 0$), then $e_X(t_2) = e_X(t_1)$.
- Wave case: For a solution of $\nabla^\alpha\nabla_\alpha u = 0$:
  - Given X^α,

$$P_X[u](t)_\alpha = \mathbf{T}[u]_{\alpha\beta}X^\beta,$$

$$E_X[u](t) = \int_{\{t'=t\}} -P_X^\alpha \mathrm{d}\nu_\alpha.$$

  - If X is (future-oriented) time-like, then $E_X \geq 0$.
  - If X is Killing, then $E_X(t_2) = E_X(t_1)$.
  - If X fails to be Killing, then $E_X(t_2) - E_X(t_1) = \int_{t_1}^{t_2}\int \nabla^\alpha X^\beta \mathbf{T}_{\alpha\beta}\mathrm{d}^4\mathrm{vol}.$

Minkowski example

In $(\mathbb{R}^{1+3}, -\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2)$,
Let

$$T = \frac{\partial}{\partial t} = \partial_t.$$

Killing, timelike.
Conserved positive energy:

$$E_T[u](t) = \int_{\{t'=t\}} \left(|\dot{u}|^2 + \sum_{i=1}^{3} |\partial_i u|^2 \right) \mathrm{d}^3x.$$

Minkowski example (cont)

Conformal (Morawetz) vector-field

$$K = (t^2 + r^2)\partial_t + 2tr\partial_r,$$
$$E_K = \int \left((t \pm r)^2 \left((\partial_t \pm \partial_r)u\right)^2 + (t^2 + r^2)|\not\nabla u|^2\right) \mathrm{d}^3x.$$

Conserved
Like $t^2 E_T$ for $|r| < (1 - \epsilon)t$.
Gives "ripple" decay.

Minkowski example (cont)

(local-decay) Morawetz vector-field

$$A = \partial_r ,$$

$$\int_{t_1}^{t_2} \int \left(\frac{1}{r^3} | \not\nabla u|^2 + \frac{1}{r^4} |u|^2 \right) \mathrm{d}^3 x \mathrm{d} t = E_A(t_2) - E_A(t_1)$$
$$\leq 2E_T .$$

Regularise: $\partial_r \mapsto \langle r \rangle^{-1} x^i \partial_i$,
denominators have $r \mapsto \langle r \rangle$,
gain $\langle r \rangle^{-2} |\partial_r u|^2$.

Norm-strengthening vector-fields

If X is Killing and $\nabla^\alpha\nabla_\alpha u = 0$, then $\nabla^\alpha\nabla_\alpha (Xu) = 0$.
If, in addition, T is time-like and Killing, then $E_T[Xu]$ is positive, conserved, and controlling (some) second derivatives.
Let $\mathbb{S}_1$ be a set of linearly independent, Killing vectors

$$\mathbb{S}_1 = \{X_1, \ldots, X_m\},$$

then let

$$\begin{aligned}
\mathbb{S}_0 =& \{\mathrm{Id}\}, \\
\mathbb{S}_n =& \mathbb{S}_1^n, \\
E_{T,n}[u] =& \sum_{i=0}^{n-1} \sum_{S\in\mathbb{S}_i} E_T[Su].
\end{aligned}$$

Example

In $\mathbb{R}^{1+3}$, $\mathbb{S}_1 = \{\partial_t, \partial_1, \partial_2, \partial_3\}$.

$$E_{T,n} \sim \int |\nabla^n u|^2 \mathrm{d}^3 x \geq \|u\|^2_{H^n(\{t\}\times\mathbb{R}^3)}.$$

Sobolev estimate: $\forall \vec{x} \in \mathbb{R}^3 : |\psi(\vec{x})| \lesssim \|\psi\|_{H^n}$ for $n > 3/2$.

$\therefore \forall (t, \vec{x}) : |u(t, \vec{x})| \lesssim E_{T,2}[u](0)^{1/2}$.
Similarly: $\forall \vec{x} : \exists C : \forall t \in \mathbb{R}$, if $\nabla^\alpha \nabla_\alpha u = 0$, then

$$|u(t, \vec{x})| \leq C t^{-1} E_{K,2}[u](0)^{1/2}.$$

Kerr spacetime

The Kerr metric

$$\begin{aligned}\mathbf{g} =& -(1-\frac{2Mr}{\Sigma})\mathrm{d}t^2 - \frac{4aMr}{\Sigma}\mathrm{d}t\mathrm{d}\phi - \frac{\Pi}{\Sigma}\sin^2\theta\mathrm{d}\phi^2 \\ &+ \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2, \\ \Delta =& r^2 - 2Mr + a^2, \\ \Sigma =& r^2 + a^2\cos^2\theta \\ \Pi =& (r^2+a^2)^2 - a^2\sin^2\theta\Delta.\end{aligned}$$

Mass M, angular momentum aM.
Schwarzschild $a = 0$.
Exterior: $(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times S^2$, $r_+ = M + \sqrt{M^2 - a^2}$.
Symmetries: ∂_t, ∂_ϕ

Hidden symmetries

For a particle
Symmetries generate: $e = \mathbf{g}(\dot\gamma, \partial_t) = \dot\gamma_t$, $l_z = \mathbf{g}(\dot\gamma, \partial_\phi) = \dot\gamma_\phi$.
Additional conserved quantity (Carter constant):

$$\mathcal{Q} =(\dot\gamma_\theta)^2 + \cot^2\theta(\dot\gamma_\phi)^2 + a^2\sin^2\theta(\dot\gamma_t)^2$$

$$\begin{aligned}\frac{1}{\Sigma}\frac{\mathrm{d}r}{\mathrm{d}\lambda} =&\sqrt{-\mathcal{R}},\\ \mathcal{R} =&\mathcal{R}(r; M, a; e, l_z, \mathcal{Q})\\ =&-(r^2+a^2)^2e^2 - 4aMrel_z + (\Delta - a^2)l_z^2 + \Delta\mathcal{Q}.\end{aligned}$$

Orbits at double roots of $\mathcal{R}$ (or of $\tilde{\mathcal{R}} = z\mathcal{R}$).
For $a = 0$, at $r = 3M$.
For $|a| \ll M$, range near $r = 3M$.

Wave in Kerr

$$\mathcal{Q} =\frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \cot^2\theta\partial_\phi^2 + a^2\sin^2\theta\partial_t^2,$$
$$\Sigma\nabla^\alpha\nabla_\alpha u =(\partial_r\Delta\partial_r + \mathcal{R}(r; M, a; \partial_t, \partial_\phi, \mathcal{Q}))u = 0.$$

Problems:

- No positive, conserved energy for $a \neq 0$.
- Trapping (at different r when $a \neq 0$).
- Not enough Killing vectors to get a stronger norm controlling spatial Sobolev norms.

Solutions: Higher energies

Using hidden symmetries for higher norms.
Let

$$\begin{aligned}\mathbb{S}_0 =&\{\mathrm{Id}\},\\ \mathbb{S}_1 =&\{\partial_t, \partial_\phi\},\\ \mathbb{S}_2 =&\{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, \mathcal{Q}\}\\ \ldots\end{aligned}$$

Given X, let

$$E_{X,n}[u] = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}_i} E_X[Su].$$

If E_X controlled radial derivatives,
then $E_{X,3}(t)$ would control

$$\int\int |\not\nabla^2 \partial_r u|^2 \mathrm{d}r \sin\theta \mathrm{d}\phi \mathrm{d}\theta \gtrsim \int |\partial_r u|^2 \mathrm{d}r$$
$$\gtrsim |u(t,r,\theta,\phi)|^2.$$

Solutions: Energy

$T_\infty = \partial_t$ is time-like for r large.
$T_{\mathcal{H}} = \partial_t + \Omega_{\mathcal{H}}\partial_\phi$ with $\Omega_{\mathcal{H}} = a/(r_+^2 + a^2)$ is time-like for $r > r_+$ but not too large.
Let

$$T_{\text{Blend}} = \partial_t + \Omega_{\mathcal{H}}\chi\partial_\phi$$

χ goes from 1 to 0 in $[R, R+M]$, $R = 2(3M)$.
For $|a|$ small, T_{Blend} is time-like everywhere.
$E_{T_{\text{Blend}}}[u](t)$ is positive.

$$E_{T_{\text{Blend}}}(t_2) - E_{T_{\text{Blend}}}(t_1) \leq \int_{t_1}^{t_2}\int \chi_{r\sim R}|\partial_r u||\partial_\phi u|\mathrm{d}^4\text{vol}.$$

Solutions: Morawetz for trapping $a = 0$

Morawetz:

$$A \sim \frac{\Delta}{r^2 + a^2} g \partial_r,$$

$$E_A(t_2) - E_A(t_1) \sim \int\int (\partial_r g)(\partial_r u)^2 - (ug(\partial_r \tilde{\mathcal{R}})u) + (\text{weight})u^2 \quad d^4\text{vol}.$$

Want:

- $\partial_r g \geq 0$,
- $g(\partial_r \tilde{\mathcal{R}})$ positive functions times elliptic operators,
- g bounded, so that $|E_A| \lesssim E_T$.

In Schwarzschild, $\partial_r \tilde{\mathcal{R}} \sim (r - 3M) r^{-4} \not\nabla \cdot \not\nabla$

take g increasing, changing sign at $3M$, bounded.

$\implies$ Morawetz-type estimate.

Second-order symmetry expansion

Let $\mathbb{S}_2 = \{S_{\underline{a}}\}$.
Let

$$\begin{aligned}\mathcal{L} =&\partial_t^2 + \partial_\phi^2 + \mathcal{Q} = \mathcal{L}^{\underline{a}} S_{\underline{a}},\\ \mathcal{R} =&\mathcal{R}^{\underline{a}} S_{\underline{a}}.\end{aligned}$$

Strengthening the vector-field method with hidden symmetries

$$\mathbf{T}[u,v] = \frac{1}{4}\left(\mathbf{T}[u+v] - \mathbf{T}[u-v]\right)$$

If $\nabla^\alpha\nabla_\alpha u = 0 = \nabla^\alpha\nabla_\alpha v$, then $\nabla^\alpha \mathbf{T}_{\alpha\beta} = 0$.
For a symmetric, double-indexed set of vector-fields $X = \{X^{\underline{ab}\beta}\}$

$$\begin{aligned} P_X[u]_\alpha =& \mathbf{T}[S_{\underline{a}}u, S_{\underline{b}}u]_{\alpha\beta}X^{\underline{ab}\beta}, \\ E_X =& \int -P_X^\alpha \mathrm{d}\nu_\alpha, \\ E_X(t_2) - E_X(t_1) =& \int\int (\nabla^\alpha X^{\underline{ab}\beta})\mathbf{T}[S_{\underline{a}}u, S_{\underline{b}}u]_{\alpha\beta}\mathrm{d}^4\mathrm{vol}. \end{aligned}$$

Morawetz

Let

$$A^{\underline{ab}} = \mathcal{L}^{(\underline{a}} \left(\partial_r \tilde{\mathcal{R}}^{\underline{b})} \right) \partial_r .$$

Morawetz identity:

$$\int\int (\partial_r u) \mathcal{L}(\partial_r^2 \tilde{\mathcal{R}})(\partial_r u) - u\mathcal{L}(\partial_r \tilde{\mathcal{R}})(\partial_r \tilde{\mathcal{R}})u + (\text{weight})u^2 \lesssim E_A(t_2) - E_A(t_1).$$

$$C \int\int \left(\frac{\Delta^2}{(r^2+a^2)^2} \frac{|\not\nabla^2 \partial_r u|^2}{r^2} + \chi_{r \not\approx 3M} \frac{|\not\nabla^3 u|^2}{r^3} \right) \mathrm{d}^4\mathrm{vol} \leq E_{T_{\text{Blend}},3}(t_2) + E_{T_{\text{Blend}},3}(t_1)$$

Closing the energy estimate

$$
\begin{aligned}
E_{T_{\text{Blend}},3}(t_2) - E_{T_{\text{Blend}},3}(t_1) \lesssim& a \int\int (\text{localised third derivatives}) \mathrm{d}^4\text{vol} \\
\lesssim& a \int\int (\text{Morawetz terms}) \mathrm{d}^4\text{vol} \\
\lesssim& aC \left(E_{T_{\text{Blend}},3}(t_2) + E_{T_{\text{Blend}},3}(t_1) \right), \\
E_{T_{\text{Blend}},3}(t_2) \lesssim& \frac{1+aC}{1-aC} E_{T_{\text{Blend}},3}(t_1).
\end{aligned}
$$

Uniformally bounded energy.

The K estimate

Let r_* be such that $\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{\Delta}{r^2+a^2}$.
$r_* \in (-\infty, \infty)$ when $r \in (r_+, \infty)$.
Let

$$K = (t^2 + r_*^2) T_{\mathrm{Blend}} + 2tr_* \left(\frac{\Pi}{(r^2+a^2)^2} \right) \partial_{r_*},$$

$$E_K(t_2) - E_K(t_1) \lesssim a \int\int t^2 \chi_{r\sim R} |\partial_r u||\partial_\phi u| \mathrm{d}^4\mathrm{vol} + \int\int t(\text{other derivatives})\mathrm{d}^4\mathrm{vol}.$$

Closing the K estimate

$$\begin{aligned}
E_{K,3}(t) - E_{K,3}(0) \lesssim & a \int_0^t \int t'^2 (\text{third derivatives}) \mathrm{d}^4\text{vol} \\
\lesssim & Ca \int_0^t \int t'^2 (\text{Morawetz terms}) \mathrm{d}^3 x \mathrm{d}t' \\
\lesssim & Cat^2 \int_0^t \int (\text{Morawetz terms}) \mathrm{d}^3 x \mathrm{d}t' \\
& + Ca \int_0^t t' \int_0^{t'} \int (\text{Morawetz terms}) \mathrm{d}^3 x \mathrm{d}t'' \mathrm{d}t' \\
\lesssim & Cat^2 E_A(t) + Ca \int_0^t t' E_A(t') \mathrm{d}t' \\
\lesssim & CaE_{K,3}(t) + Ca \int_0^t (t')^{-1} E_{K,3} \mathrm{d}t'.
\end{aligned}$$

Roughly:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_{K,3}(t) - Cat^{-1}E_{K,3}(t) &\leq 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-Ca}E_{K,3}(t)\right) &\leq 0\\
E_{K,3}(t) &\leq t^{Ca}E_{K,3}(0).
\end{aligned}
$$

Local energy (with extra derivatives) like t^{-2+Ca}.
For fixed (r, θ, ϕ),

$$
|u(t, r, \theta, \phi)| \leq Ct^{-1+Ca}\left(E_{K,3}(0)^{1/2} + E_{T_{\text{Blend}},7}(0)^{1/2}\right)
$$

Conclusion

- Use T_{Blend} to get positive energy.
- Use classical and hidden symmetries to get higher derivative energies.
- Replace vector-fields by double-indexed sets of vector-fields and $\mathbf{T}[u]$ by $\mathbf{T}[S_{\underline{a}}u, S_{\underline{b}}u]$.
- Construct a Morawetz "vector-field" adapted to Kerr orbitting geodesics.
- Close T_{Blend} and K estimates.
- Bounded energy
- $|u| \lesssim t^{-1+Ca}$.

Likely questions

- Near and far decay 28
- Hardy 29

Let $v_\pm = t \pm r_*$.
For $r < 3M$,

$$|u| \leq v_+^{-1+Ca} |||u(0)|||.$$

For $r > 3M$,

$$|u| \leq v_-^{-1/2+Ca} r^{-1} |||u(0)|||.$$

The term to control $\chi_{r\sim R}|\partial_r u||\partial_\phi u|$ is only in a stationary region, advance in t as far as needed, then deform the surfaces to be null from $r \sim R$.
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Hardy

Morawetz is like

$$\begin{aligned} &\frac{\Delta^2}{(r^2+a^2)^2}(\partial_r u)^2 \\ &+(\text{angular terms}) \\ &+\frac{9r^2-46Mr+54M^2+O(a^2)}{r^4}u^2. \end{aligned}$$

Relate to an ODE (cf. Soffer) and relate to an ODE.
First corrections used spherical harmonic decomposition, had g change sign at peak of effective potential, so sum of angular and lower-order terms was positive.
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