

Hidden symmetries and decay for the wave equation outside a Kerr black hole

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Oberwolfach

Joint with Lars Andersson.

- ▶ Kerr spacetime
 - ▶ parameters: M mass and Ma angular momentum.
 - ▶ rotating black hole, expected end state.
 - ▶ black hole for $|a| \leq M$; $a = 0$ is Schwarzschild.
- ▶ Wave: $\nabla^\alpha \nabla_\alpha \psi = 0$, decoupled, important equation, model.
- ▶ Goal: robust tools for community.
- ▶ We consider $|a| \ll M$, exterior $r > r_+$.
- ▶ Result: $t^{-1+|a|C}$ decay for $|a| \ll M$ [arXiv:0908.2265].

- ▶ $|a| \leq M$, mode decay: Finster-Kamran-Smoller-Yau
- ▶ $|a| \ll M$: Dafermos-Rodnianski, Tataru-Tohaneanu
- ▶ (Builds on earlier Schwarzschild Morawetz and conformal energy results: Łaba- Soffer, B- Soffer, B- Sterbenz, Dafermos- Rodnianski, Metcalfe- Marzuola- Tataru-Tohaneanu, Luk.)
- ▶ spectral and scattering results.

Energy momentum tensor

Energy-momentum tensor:

$$T[\psi]_{\alpha\beta} = \nabla_{\alpha}\psi\nabla_{\beta}\psi - g_{\alpha\beta}(\nabla_{\gamma}\psi\nabla^{\gamma}\psi).$$

$$P_X[\psi]_{\alpha} = T[\psi]_{\alpha\beta}X^{\beta},$$

$$E_X[\psi](\Sigma) = \int_{\Sigma} P_X[\psi]_{\alpha}d\nu^{\alpha}.$$

Properties:

1. \mathbf{T} timelike, $\implies E_{\mathbf{T}} \geq 0$.

2. $E_X(t_2) - E_X(t_1) = \int T[\psi]_{\alpha\beta}\nabla^{(\alpha}X^{\beta)}d^4\mu_g$.

3. S a symmetry, \implies Same properties for $E_X[S\psi]$.

(generalised symmetry S : $\nabla^{\alpha}\nabla_{\alpha}\psi = 0$ gives $\nabla^{\alpha}\nabla_{\alpha}S\psi = 0$)

Minkowski example

$$\mathbf{T} = \partial_t,$$

$$\mathbf{K} = (t^2 + r^2)\partial_t + 2tr\partial_r,$$

$$\mathbf{A} = \partial_r.$$

$$E_{\mathbf{T}} = \int (\partial_t \psi)^2 + |\nabla \psi|^2 d^3x,$$

$$E_{\mathbf{K}} = \int |(t \pm r)(\partial_t \pm \partial_r)\psi|^2 + (t^2 + r^2)r^{-2} |\nabla \psi|^2 d^3x,$$

$$|E_{\mathbf{A}}| \leq E_{\mathbf{T}},$$

$$E_{\mathbf{A}}(t_2) - E_{\mathbf{A}}(t_1) = \int_{t_1}^{t_2} \int r^{-3} |\nabla \psi|^2 d^3x + |\psi(t, 0)|^2 dt.$$

Graded symmetry algebra:

$$\mathbb{S}_0 = \{\text{Id}\},$$

$$\mathbb{S}_1 = \{\text{Killing fields}\},$$

$$\mathbb{S}_n = \mathbb{S}_1^n,$$

$$E_{X,n+1}[\psi] = \sum_{i=0}^n \sum_{S \in \mathbb{S}_i} E_X[S\psi].$$

In Minkowski: $E_{T,n}$ controls the H^n norm squared.
(Away from $r = 0$, rotations sufficient.)

Features of Kerr

- ▶ Boyer-Lindquist coordinates: exterior
 $(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times S^2$.
- ▶ $ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$
- ▶ **Ergo-region:** ∂_t spacelike.
- ▶ $g_{ij} = g_{ij}(r, \theta)$: Symmetries: $\partial_t, \partial_\phi$.
- ▶ Conserved quantities for null geodesics:
 $g(\dot{\gamma}, \dot{\gamma}), \dot{\gamma}_t, \dot{\gamma}_\phi, Q = \dot{\gamma}_\theta^2 + \cot^2 \theta \dot{\gamma}_\phi^2 + a^2 \sin^2 \theta \dot{\gamma}_t^2$.

$$\frac{1}{\Sigma^2} \left(\frac{dr}{d\lambda} \right)^2 = -\mathcal{R}(r; M, a; \dot{\gamma}_t, \dot{\gamma}_\phi, Q).$$

- ▶ **Photon orbits** bifurcate from $3M$.

Problems:

1. No timelike, Killing vector: no positive, conserved energy.
2. Not enough Killing vectors for higher energies.
3. Photon orbits: “trapping”.
4. r^{-3} effective potential.
5. No timelike, Killing vector (for \mathbf{K}).
6. Principle null vectors are not integrable (in Frobenius sense).

Blended timelike vector field

- ▶ Stationary vector field timelike for r large ∂_t .
- ▶ Null generator extension timelike for r near r_p $\partial + \omega_H \partial_\phi$.
- ▶ For $|a|$ small overlap.

Let

$$T_\chi = \partial_t + \chi \omega_H \partial_\phi.$$

χ goes from 1 to 0 in $[10M, 11M]$.

- ▶ globally timelike
- ▶ Killing outside $r \in [10M, 11M]$
- ▶ but degenerates on horizon.

Almost conservation

Assume higher norm defined, and

$$|E_{\mathbf{A},3}| \leq E_{\mathbf{T},3}$$
$$E_{\mathbf{A},3}(t_2) - E_{\mathbf{A},3}(t_1) \geq \int \int (\text{weights}) |\partial^3 \psi|^2 d^3 x dt.$$

then, since

$$E_{T_x,3}(t_2) - E_{T_x,3}(t_1) \geq |a| C \int_{t_1}^{t_2} \int_{10M}^{11M} \int_{S^2} |\partial^2 \partial_r \psi| |\partial^2 \partial_\phi \psi| d^2 \omega dr dt.$$

if $|a|$ small

$$E_{T_x,3}(t_2) \leq \frac{1 + C|a|}{1 - C|a|} E_{T_x,3}(t_1).$$

Hidden symmetry from Carter Killing 2-tensor

$$Q = \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{\cos^2 \theta}{\sin \theta} \theta \partial_\phi^2 + a^2 \sin^2 \theta \partial_t^2.$$

Symmetry algebra

$$\mathbb{S}_0 = \{\text{Id}\},$$

$$\mathbb{S}_1 = \{\partial_t, \partial_\phi\},$$

$$\mathbb{S}_2 = \{\partial_t^2, \partial_t \partial_\phi, \partial_\phi^2, Q\},$$

$$\mathbb{S}_n = \{\partial_t^{n_t} \partial_\phi^{n_\phi} Q^{n_Q} \mid n_t + n_\phi + 2n_Q = n\}.$$

Higher norm from \mathbb{S}_n .

Pointwise estimate

(Away from horizon:)

Since $\Delta + a^2 \sin^2 \theta \partial_t^2 = Q + \partial_\phi^2$,

$$\begin{aligned} \int_{S^2} |\Delta \psi|^2 \sin \theta d\theta d\phi &\leq C \int_{S^2} (|\partial_t^2 \psi|^2 + |\partial_\phi^2 \psi|^2 + |Q\psi|^2) \sin \theta d\theta d\phi \\ &\leq C \int |\partial_r \psi|_2^2 r^2 dr \\ &\leq CE_{T_x, 3} \end{aligned}$$

By spherical Sobolev

$$|\psi(t, r, \theta, \phi)| \leq CE_{T_x, 3}^{1/2}$$

The conformal energy

Introduce r_*

$$\frac{dr}{dr_*} = \frac{r^2 - 2MR + a^2}{r^2 + a^2},$$
$$u_{\pm} = t \pm r_*.$$

Let

$$\mathbf{K} = (t^2 + r_*^2) T_{\chi} + 2\tilde{N} t r_* \partial_{r_*},$$
$$\tilde{N} = \frac{(r^2 + a^2)^2}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}.$$

\tilde{N} is used to cancel worst term in deformation.

Lower-order term for second worst.

The conformal energy (part 2)

For $E_{\mathbf{K},3}$, worst remaining term in deformation like

$$|a|t^2 \mathbb{1}_{\text{supp}\chi'} |\partial_r \psi|_2 |\partial_\phi \psi|_2.$$

Morawetz estimate gives roughly a gain of t^{-1} for integrability.

$$\begin{aligned} E_{\mathbf{K},3}(T) - E_{\mathbf{K},3}(1) &\leq |a|C \int_1^T t^{-1} E_{\mathbf{K},3}(t) dt + \dots, \\ &\sim \frac{d}{dt} E_{\mathbf{K},3} = |a|Ct^{-1} E_{\mathbf{K},3} \\ E_{\mathbf{K},3} &\leq t^{|a|C'} CE_{\mathbf{K},3}(1) \\ &\quad + CE_{T_\chi,7}. \end{aligned}$$

Pointwise decay (stationary decay)

For r fixed, $\mathbf{K} \sim t^2 T_\chi$, so local energy decays like $t^{-2+|a|}C$.

$$|\psi(t, r, \theta, \phi)| \leq t^{-1+|a|C'} C (E_{\mathbf{K},3}(1) + E_{T_\chi,7})$$

Pointwise decay (horizon and null infinity)

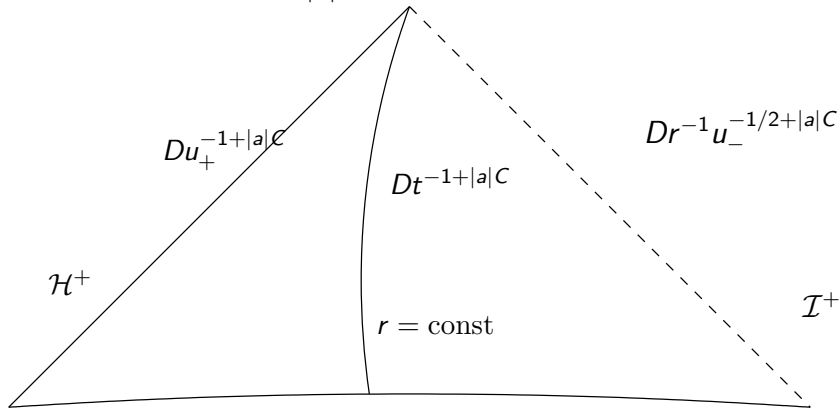
$$\mathbf{K} \neq u_+^2 L_+ + u_-^2 L_-.$$

No \mathbf{K} -compatible null foliation.

- ▶ Near \mathcal{I}^+ , use $t \sim r_*(1 + Cr_*^{-1})$ near \mathcal{I}^+ .
- ▶ Near \mathcal{H}^+ , on (u_+, u_-) quadrilateral bounded by constant u_+ , u_- , $t = 0$, apply Green's theorem with $(\partial_{u_-} \psi) du_-$.

Effective transport equations: $t^{-1+|a|C}$ equivalent extends to entire manifold.

Theorem: On Kerr with $|a| \ll M$,



Initial data

$$D^2 = \|\psi\|(0)^2 = E_{T_{x,9}}(0) + E_{K,5}(0) + E_{n,3}(0).$$

??

Bilinear energy-momentum and \mathbb{S}_2 vector fields.

Recall:

$$\mathbb{S}_2 = \{\partial_t^2, \partial_t \partial_\phi, \partial_\phi^2, Q\} = \{\underline{S}_a\}.$$

Let

$$\begin{aligned} T[\psi_1, \psi_2]_{\alpha\beta} &= (1/4) (T[\psi_1 + \psi_2]_{\alpha\beta} - T[\psi_1 - \psi_2]_{\alpha\beta}), \\ T[\psi_1]_{\underline{ab}\alpha\beta} &= T[\underline{S}_a \psi, \underline{S}_b \psi]_{\alpha\beta}. \end{aligned}$$

Given $X^{\underline{ab}}$, let

$$\begin{aligned} P_{X^{\underline{ab}}}[\psi]_\alpha &= T[\psi_1]_{\underline{ab}\alpha\beta} X^{\underline{ab}\beta}, \\ E_{X^{\underline{ab}}}[\psi] &= \int P_{X^{\underline{ab}}}[\psi]_\alpha d\nu^\alpha. \end{aligned}$$

Morawetz estimate

Wave equation takes form $(\partial_r \Delta \partial_r - \Delta^{-1} \mathcal{R}(r)^a S_a) \psi = 0$.
($\Delta = r^2 - 2Mr + a^2$)

Morawetz should give dispersion about orbiting geodesics.

Build A^{ab} from \mathcal{R}^a , \mathcal{L}^b , $z(r)$, $w(r)$, where

$$\mathcal{L}^a S_a = \partial_t^2 + \partial_\phi^2 + Q.$$

Suitable choices $\implies E_{T_{\chi,3}} \geq CE_{A^{ab}}$.
(Lower-order terms)

Morawetz estimate (cont.)

Get $T_{ab\alpha\beta}\nabla^\alpha A^{ab\beta}$ like

$$\begin{aligned} & \Delta^{3/2} z^{1/2} \left(\partial_r w \frac{z^{1/2}}{\Delta^{1/2}} \left(-\partial_r \frac{z}{\Delta} \mathcal{R}^a \right) \right) (\partial_r S_{\underline{a}} \psi) (\partial_r S_{\underline{b}} \psi) \\ & + w \left(\partial_r \frac{z}{\Delta} \mathcal{R}^a \right) \left(\partial_r \frac{z}{\Delta} \mathcal{R}^b \right) \mathcal{L}^{\alpha\beta} (\partial_\alpha S_{\underline{a}} \psi) (\partial_\beta S_{\underline{b}} \psi) \\ & + \frac{1}{4} \left(\partial_r \Delta \partial_r z \left(\partial_r w \left(\partial_r \frac{z}{\Delta} \mathcal{R}^b \right) \right) \right) \mathcal{L}^b (S_{\underline{a}} \psi) (S_{\underline{b}} \psi). \end{aligned}$$

So

$$\begin{aligned} & E_{T_x,3}(t_2) + E_{T_x,3}(t_1) \\ & \geq C \int \frac{1}{r^2} |\partial_r \psi|_2^2 + \mathbb{1}_{r \neq 3M} \frac{1}{r^3} (|\partial_t \psi|_2^2 + |\nabla \psi|_2^2) + \frac{1}{r^4} |\psi|_2 d^4 \mu_g. \end{aligned}$$

Morawetz is like

$$\begin{aligned} & \frac{\Delta^2}{(r^2 + a^2)^2} (\partial_r u)^2 \\ & + (\text{angular terms}) \\ & + \frac{9r^2 - 46Mr + 54M^2 + O(a^2)}{r^4} u^2. \end{aligned}$$

Relate to an ODE (cf. Soffer); need positive solutions.

Explicit ODE solution by hypergeometric functions.

Limits range of a .

Open Problems

- ▶ $0 \leq |a| < M$.
- ▶ decay rate: t^{-1} , $t^{-3/2}$, $t^{-3+\epsilon}$, t^{-3} .
- ▶ Maxwell.
- ▶ linearised gravity.
- ▶ Spacetimes with similar symmetry structure: Kerr-ADS, Kerr-Newman
- ▶ Spacetimes with asymptotic symmetries: (Spherical Einstein-scalar field plus additional decoupled scalar field).
- ▶ Vector-field proof of spherical Einstein-scalar field results.
- ▶ (Numerically) understand evolution of a , M , $\vec{\omega}$, \vec{v} .
- ▶ Kerr stability.