Hidden symmetries and decay for the wave equation outside a Kerr black hole

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Overview

- Kerr: rotating black hole, expected end state.
- parameters: M mass and Ma angular momentum.
- black hole for $|a| \leq M$.
- $a = 0$ is Schwarzschild.
- We consider $|a| \ll M$, exterior $r > r_+$.
- Wave: $\nabla^\alpha \nabla_\alpha \psi = 0$, decoupled, important equation, model.
- Joint with Lars Andersson.

Previous results

- $|a| \leq M$, mode decay: Finster-Kamran-Smoller-Yau
- $|a| \ll M$: Dafermos-Rodnianski, Tataru-Tohaneanu
- (Builds on earlier Schwarzschild Morawetz and conformal energy results: Łaba- Soffer, B- Soffer, B- Sterbenz, Dafermos- Rodnianski, Metcalfe- Marzuola- Tataru- Tohaneanu, Luk.)
- spectral and scattering results.

Energy-momentum tensor

Energy-momentum tensor:

$$T[\psi]_{\alpha\beta} = \nabla_\alpha\psi\nabla_\beta\psi - g_{\alpha\beta}(\nabla_\gamma\psi\nabla^\gamma\psi).$$

Given a vector-field X, 4-momentum

$$P_X[\psi]_\alpha = T[\psi]_{\alpha\beta}X^\beta,$$
$$E_X[\psi](\Sigma) = \int_\Sigma P_X[\psi]_\alpha \mathrm{d}\nu^\alpha.$$

Assume spacetime is foliated by Σ_t with timelike, future-oriented normal

$$E_X[\psi](\Sigma_t) = E_X[\psi](t) = E_X[\psi] = E_X(t)$$

Energy-momentum properties

Properties:

1. If $\mathbf{T}$ timelike,
   then $E_{\mathbf{T}} \geq 0$.

2. If X is Killing,
   then $E_X(t_2) = E_X(t_1)$.
   Otherwise: $E_X(t_2) - E_X(t_1) = \int T[\psi]_{\alpha\beta}\nabla^{(\alpha}X^{\beta)}\mathrm{d}^4\mu_g$.

3. We call S a (generalised) symmetry when
   $\nabla^\alpha\nabla_\alpha\psi = 0 \implies \nabla^\alpha\nabla_\alpha S\psi = 0$,
   If S is a symmetry,
   then $E_X[S\psi]$ has the same properties as $E_X[\psi]$.

Problems:

1. No positive conserved energy
2. Lack of sufficient symmetries for Sobolev estimates
3. Complicated orbiting geodesics, filling an open set.

Symmetry algebra

The set of symmetries forms a graded algebra.
Iterated Lie derivatives along Killing vectors are symmetries.
In Kerr, there are Killing vectors

$$\partial_t, \qquad \partial_\phi.$$

There is also the (hidden) symmetry:

$$\mathrm{Q} = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{\cos^2\theta}{\sin^2\theta}\partial_\phi^2 + a^2\sin^2\theta\partial_t^2.$$

Order n generators:

$$\mathbb{S}_n = \{\partial_t^{n_t}\partial_\phi^{n_\phi}\mathrm{Q}^{n_Q} \mid n_t, n_\phi, n_Q \in \mathbb{N}, n_t + n_\phi + 2n_Q = n\},$$
$$\mathbb{S}_0 = \{\mathrm{Id}\}, \ \ \mathbb{S}_1 = \{\partial_t, \partial_\phi\}, \ \ \mathbb{S}_2 = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, \mathrm{Q}\} = \{S_{\underline{a}}\}.$$

Higher energies and momenta for $\mathbb{S}_2$ vectors

Let

$$E_{X,n+1}[\psi] = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}_i} E_X[S\psi],$$

$$|\psi|_n^2 = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}_i} |S\psi|^2.$$

Let

$$T[\psi_1,\psi_2]_{\alpha\beta} = (1/4)\left(T[\psi_1+\psi_2]_{\alpha\beta} - T[\psi_1-\psi_2]_{\alpha\beta}\right),$$

$$T[\psi_1]_{\underline{ab}\alpha\beta} = T[S_{\underline{a}}\psi, S_{\underline{b}}\psi]_{\alpha\beta}.$$

Given $X^{\underline{ab}}$, let

$$P_{X^{\underline{ab}}}[\psi]_\alpha = T[\psi_1]_{\underline{ab}\alpha\beta} X^{\underline{ab}\beta},$$

$$E_{X^{\underline{ab}}}[\psi] = \int P_{X^{\underline{ab}}}[\psi]_\alpha \mathrm{d}\nu^\alpha.$$

The blended energy

∂_t is timelike for r large
$\partial_t + \omega_H \partial_\phi$ (with $\omega_H = a/(r_+^2 + a^2)$) is timelike for r near r_+.
Let

$$T_\chi = \partial_t + \chi \omega_H \partial_\phi .$$

$$E_{T_\chi, n+1}(t_2) - E_{T_\chi, n+1}(t_1) \leq |a| C \int_{t_1}^{t_2} \int \mathbb{1}_{\text{supp}\chi'} |\partial_r \psi|_n |\partial_\phi \psi|_n \mathrm{d}^4 \mu_g$$

Morawetz estimate

Wave equation takes form $\left(\partial_r \Delta \partial_r - \Delta^{-1}\mathcal{R}(r)^{\underline{a}} S_{\underline{a}}\right)\psi = 0$.
$(\Delta = r^2 - 2Mr + a^2)$

Build $A^{\underline{ab}}$ from $\mathcal{R}^{\underline{a}}$, $\mathcal{L}^{\underline{b}}$, and weights z and w, where

$$\mathcal{L}^{\underline{a}} S_{\underline{a}} = \partial_t^2 + \partial_\phi^2 + \mathrm{Q}.$$

Suitable choices $\implies$ $E_{T_\chi,3} \geq C E_{A^{\underline{ab}}}$.
(Lower-order terms)

Morawetz estimate (cont.)

Get $T_{\underline{ab}\alpha\beta}\nabla^\alpha A^{\underline{ab}\beta}$ like

$$\Delta^{3/2}z^{1/2}\left(\partial_r w\frac{z^{1/2}}{\Delta^{1/2}}\left(-\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\right)(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)$$
$$+w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)$$
$$+\frac{1}{4}\left(\partial_r\Delta\partial_r z\left(\partial_r w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\right)\right)\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).$$

So

$$E_{T_\chi,3}(t_2)+E_{T_\chi,3}(t_1)$$
$$\geq C\int\frac{1}{r^2}|\partial_r\psi|_2^2+\mathbb{1}_{r\not\approx 3M}\frac{1}{r^3}\left(|\partial_t\psi|_2^2+|\not\nabla\psi|_2^2\right)+\frac{1}{r^4}|\psi|_2\mathrm{d}^4\mu_g.$$

Bounded energy argument

$$E_{T_\chi,3}(t_2) - E_{T_\chi,3}(t_1) \geq |a| C \int ((localisation)) |\partial^3 \psi|^2 \mathrm{d}^4 \mu_g$$
$$\geq |a| C (E_{T_\chi,3}(t_2) + E_{T_\chi,3}(t_1)).$$

$$E_{T_\chi,3}(t_2) \leq \frac{1 + |a|C}{1 - |a|C} E_{T_\chi,3}(t_1)$$

Pointwise estimate

(Away from horizon:)
Since $\not\Delta + a^2 \sin^2\theta \partial_t^2 = Q + \partial_\phi^2$,

$$\begin{aligned}\int_{S^2} |\not\Delta\psi|^2 \sin\theta d\theta d\phi \leq & C \int_{S^2} \left(|\partial_t^2\psi|^2 + |\partial_\phi^2\psi|^2 + |Q\psi|^2 \right) \sin\theta d\theta d\phi \\ \leq & C \int |\partial_r\psi|_2^2 r^2 dr \\ \leq & C E_{T_\chi,3}\end{aligned}$$

By spherical Sobolev

$$|\psi(t,r,\theta,\phi)| \leq C E_{T_\chi,3}^{1/2}$$

The conformal energy

Introduce r_*

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}r_*} =& \frac{r^2 - 2MR + a^2}{r^2 + a^2}, \\ u_\pm =& t \pm r_*. \end{aligned}$$

Let

$$\begin{aligned} K =& (t^2 + r_*^2) T_\chi + 2\tilde{N} t r_* \partial_{r_*}, \\ \tilde{N} =& \frac{(r^2 + a^2)^2}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}. \end{aligned}$$

$\tilde{N}$ is used to cancel worst term in deformation.
Lower-order term for second worst.

The conformal energy (part 2)

For $E_{K,3}$, worst remaining term in deformation like

$$|a|t^2 \mathbb{1}_{\text{supp}\chi'} |\partial_r \psi|_2 |\partial_\phi \psi|_2 .$$

Morawetz estimate gives roughly a gain of t^{-1} for integrability.

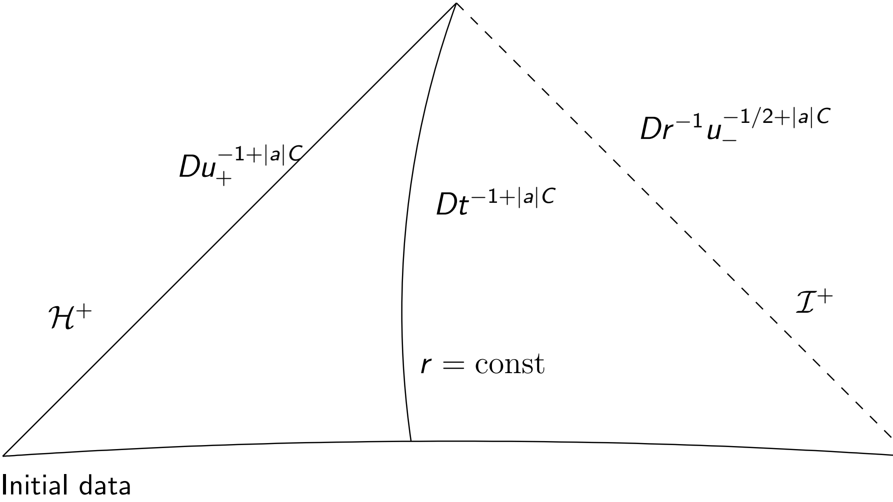
$$\begin{aligned} E_{K,3}(T) - E_{K,3}(1) \leq& |a| C \int_1^T t^{-1} E_{K,3}(t) \mathrm{d}t + \ldots , \\ E_{K,3} \leq& t^{|a|C'} C E_{K,3}(1) \\ &+ C E_{T_\chi,7} . \end{aligned}$$

Pointwise decay

For r fixed, $K \sim t^2 T_\chi$, so local energy decays like $t^{-2+|a|C}$.

$$|\psi(t,r,\theta,\phi)| \leq t^{-1+|a|C'} C \left(E_{K,3}(1) + E_{T_\chi,7} \right)$$

Theorem: On Kerr with $|a| \ll M$,



$$D^2 = \|\psi\|(0)^2 = E_{T_\chi,9}(0) + E_{K,5}(0) + E_{n,3}(0).$$