Decay for the Maxwell field outside a slowly rotating Kerr black hole

Pieter Blue
University of Edinburgh

March 2011
JAMI

Joint with L. Andersson and J.-P. Nicolas.

- Kerr spacetime
  - parameters: M mass and Ma angular momentum.
  - rotating black hole, expected end state.
  - black hole for $|a| \leq M$; $a = 0$ is Schwarzschild.
- Maxwell system: physical and model.
- Goal: robust tools (hopefully) for Kerr stability.
- We consider $|a| \ll M$, exterior $r > r_+$.
- Work in progress

General relativity in 1 slide

- $\mathcal{M}$: space-time manifold.
- g: Lorentz (-,+,+,+ signature) pseudometric.
  - Time-like vector: $g(v,v) < 0$,
  - null vector: $g(v,v) = 0$,
  - space-like vector: $g(v,v) > 0$.
  - Summation convention $g(v,v) = g_{\alpha\beta}v^\alpha v^\beta$.
  - Curve length: $\int \sqrt{|g(\dot\gamma,\dot\gamma)|}ds$.
- (Vacuum) Einstein equation:

$$\text{Ric}[g] = 0.$$

Geometry of Schwarzschild and Kerr

- Schwarzschild is $a = 0$, subcritical Kerr is $|a| < M$.
- Spherical co-ordinates, $(t, r, \theta, \phi) = (t, r, \omega)$:

$$
\begin{aligned}
g =& -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 \\
&+ \Sigma d\theta^2 + \left((r^2 + a^2)^2 - a^2 \Delta \sin^2\theta\right) \frac{\sin^2\theta}{\Sigma} d\phi^2, \\
\Sigma =& r^2 + a^2\cos^2\theta, \\
\Delta =& r^2 - 2Mr + a^2.
\end{aligned}
$$

- Exterior: $r > r_+ = M + \sqrt{M^2 - a^2}$.
- (Classical) Symmetries: ∂_t, ∂_ϕ.

Further notation

$$\Sigma_t = \{t\} \times (r_+, \infty) \times S^2.$$

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{\Delta}{r^2 + a^2}.$$

Stability and strategy

- Minkowski stability:
  future stability [Friedrich],
  Cauchy stability [Christodoulou-Klainerman, Lindblad-Rodnianski].
- Linear part by **vector-field method**
- Models:

wave	$s = 0$
Maxwell	$s = 1$
linearised gravity*	$s = 2$

Schwarzschild and Kerr wave decay results

- Scattering: Dimock-Kay, Bachelot, Nicolas, Häfner.
- Spherical wave-Einstein: Dafermos-Rodnianski, Holzegel, Dafermos-Holzegel.
- Schwarzschild decay: Łaba-Soffer, Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski, Marzoula-Metcalfe-Tataru-Tohaneanu, Holzegel, Luk, Donninger-Schlag-Soffer, Laul-Metcalfe.
- Kerr: Finster-Kamran-Smoller-Yau, Dafermos-Rodnianski, Tataru-Tohaneanu, Andersson-Blue, Tohaneanu, Tataru.

Higher spin fields in Kerr

(Dirac is very different)

- Schwarzschild scattering: Bachelot
- Schwarzschild decay: Finster-Smoller, Blue, Holzegel.
- Kerr has no exponentially divergent modes: Whiting.

The Maxwell field

$$F \in \wedge^2 \qquad F_{\alpha\beta} = F_{[\alpha\beta]}$$
$$\mathrm{d}F = 0 \qquad \nabla_{[\alpha} F_{\beta\gamma]} = 0$$
$$\mathrm{d}^* F = 0 \qquad \nabla^{\alpha} F_{\alpha\beta} = 0$$

Null basis l, n, e_A, e_B (complexify $m = e_A + ie_B$)
(Use principle null vectors in Kerr.)

$$\phi_1 = F(l, m)$$
$$\phi_0 = \frac{1}{2}(F(l, n) + F(\bar{m}, m))$$
$$\phi_{-1} = F(n, \bar{m})$$

$s = 0$: u; $s = 2$: $\psi_{-2}, \psi_{-1}, \psi_0, \psi_1, \psi_2$;

Reduced systems

$\varphi \in \{u, \phi, \psi\}$.

a	s	Equation	
0	$0, 1, 2$	$(\Box + s^2 V)\varphi_0 = 0$	Regge-Wheeler
$< M$	$0, 1, 2$	$L_{\pm s}\varphi_{\pm s} = 0$	Teukolsky
$< M$	0	$\Box u = 0$	Wave
$< M$	1	$(\Box + V)\phi_0 = 0$	Fackerel-Ipser
$< M$	2	$(\Box + 4V)\psi_0 = 0$	Aksteiner-Andersson.

Regge-Wheeler: V nonzero curvature component.
Fackerel-Ipser and AA: V complex combination of two nonzero curvature components.

Vector-field method

Energy-momentum tensor:

$$\begin{aligned} \mathrm{T}[\psi]_{\alpha\beta} &\quad \text{from varying Lagrangian}, \\ P_{(X)}[\psi]_\alpha &= \mathrm{T}[\psi]_{\alpha\beta} X^\beta, \\ E_X[\psi](\Sigma) &= \int_\Sigma P_{(X)}[\psi]_\alpha \mathrm{d}\nu^\alpha. \end{aligned}$$

Typical properties:

EG1 T future-directed, timelike, $\implies E_T \geq 0$.

EG2 $E_X(\Sigma_2) - E_X(\Sigma_1) = \int \mathrm{T}[\psi]_{\alpha\beta} \nabla^{(\alpha} X^{\beta)} \mathrm{d}^4\mu_g$.

ES S a symmetry, $\implies$ Same properties for $E_X[S\psi]$.
(generalised symmetry: φ solves PDE $\implies S\varphi$ solves PDE.)

The Killing algebra

A tensor $K_{\alpha_1\dots\alpha_n}$ is **Killing** if

$$K_{\alpha_1\dots\alpha_n} = K_{(\alpha_1\dots\alpha_n)} \quad \text{symmetric,}$$
$$\nabla_{(\alpha} K_{\alpha_1\dots\alpha_n)} = 0 \quad \text{derivative has no symmetric part.}$$

Conserved quantity along a geodesic γ:

$$(\dot\gamma^\alpha \nabla_\alpha)\left(K_{\alpha_1\dots\alpha_n}\dot\gamma^{\alpha_1}\dots\dot\gamma^{\alpha_n}\right) = 0.$$

The set forms a graded algebra.
In Kerr, the algebra is generated by ∂_t, ∂_ϕ, and
$K_{\alpha\beta} = \Delta_{S^2} + a^2\sin^2\theta\partial_t^2$.

Obstacles

- No positive, conserved energy
- Orbiting null geodesics ("trapping", also complicated)
- [Insufficient (classical) symmetries]
- Bound states

Energy and Morawetz (integrated local energy) estimate

For the Schwarzschild wave equation, let

$$E_T[\phi](t) = \int_{\Sigma_t} |\partial_t \phi|^2 + |\partial_{r_*} \phi|^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)|\nabla_{S^2}\phi|^2 \mathrm{d}^2\omega \mathrm{d}r_*.$$

Key estimates:

$$E_T[\phi](t_2) = E_T[\phi](t_1) \tag{SW:T}$$

$$\int_{t_1}^{t_2}\int_{\Sigma_t} \frac{|\partial_{r_*}\phi|^2}{1+r_*^2} + \frac{1}{r^3}\frac{(r-3M)^2}{r^2}\left(1-\frac{2M}{r}\right)\left(|\partial_t\phi|^2 + |\nabla_{S^2}\phi|^2\right) + \frac{1}{1+r_*^4} u^2 \mathrm{d}^2\omega \mathrm{d}r_* \mathrm{d}t \leq E_T[\phi](t_2) + E_T[\phi](t_1). \tag{SW:A}$$

Goal: something similar for Maxwell (after subtracting bound state).

Energy-momentum tensors

$$\mathrm{T}[F]_{\alpha\beta} = F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta},$$
$$\mathrm{T}[u]_{\alpha\beta} = \nabla_{\alpha}u\nabla_{\beta}u - \frac{1}{2}g_{\alpha\beta}\nabla_{\gamma}u\nabla^{\gamma}u.$$

Note: F energies are L^2, but u energies are H^1.

"Bird norm": After applying second-order symmetries,
F (i.e. $\phi_{-1,0,1}$) will be in H^{2n},
but ϕ_0 will be in H^{2n+1}.

Estimate I: F with T

$$T = \partial_t + \frac{a}{r^2 + a^2} \chi \partial_\phi .$$

Globally time-like (in exterior).
Fails to be Killing only on suppχ.

Difference between initial and final energy of ϕ_{-1}, ϕ_0, ϕ_1 (of order $2n$)
controlled by a times space-time integral of ($2n$ derivatives) $\phi_{-1}, \phi_0, \phi_1$.

Estimate II: F with A

$$A = f(r)\partial_r$$

Spacetime integral of ($2n$ derivatives) $\phi_{\pm 1}$
controlled by initial and final energies (of order $2n$)
plus spacetime integral of ($2n$ derivatives) ϕ_0.

Estimate III: ϕ_0 with A

The "driving component", ϕ_0, satisfies the Fackerel-Ipser equation, so we can derive estimates for this PDE.
(Replace A by a combination of second-order symmetries and the radial derivatives, so work at regularity H^3. Also include a lower-order term.)

Spacetime integral of $2n+1$ (except at null geodesic orbits) and $2n$ derivatives of ϕ_0
controlled by initial and final energy of ϕ_0 (of order $2n+1$)
plus localised spacetime integral of ($2n$ derivatives) ϕ_0.

Estimate IV: Apply algebraic topology!

The bound states are exactly the Coulomb states.
The charges can be read from the initial data as

$$\int_{S^2} {}^*F, \qquad \int_{S^2} F$$

After projecting out the bound states:
Spacetime integral of ($2n$ derivatives) ϕ_0
controlled by spacetime integral of ($2n+1$ derivatives) of ϕ_0
plus a times spacetime integral of ($2n$ derivatives) of ϕ_{-1} and ϕ_1.

Summary

$\tilde{\chi}$ vanishes at trapping.

$$E_{2n}[F](t) - E_{2n}[F](0) \leq |a| \int (\text{supp}\chi)|\partial^{2n}\phi_{-1,0,1}|^2$$

$$\int |\partial^{2n}\phi_{-1,1}|^2 \leq E_{2n}[F](t) + E_{2n}[F](0) + \int |\partial^{2n}\phi_0|^2$$

$$\int \tilde{\chi}|\partial^{2n+1}\phi_0|^2 + |\partial^{2n}\phi_0|^2 \leq E_{2n+1}[\phi_0](t) + E_{2n+1}[\phi_0](0) + \int |\partial^{2n}\phi_0|^2$$

$$\int \tilde{\chi}|\partial^{2n}\phi_0|^2 \leq \int \tilde{\chi}|\partial^{2n+1}\phi_0|^2 + |a| \int \tilde{\chi}|\partial^{2n}|\phi_{-1,1}|^2$$

$$E_{2n+1}[\phi_0](t) - E_{2n+1}[\phi_0](0) \leq |a| \int (\text{supp}\chi)|\partial^{2n+1}\phi_0|^2$$

$$E_{2n}[F](t) - E_{2n}[F](0) + E_{2n+1}[\phi_0](t) - E_{2n+1}[\phi_0](0)$$
$$+ \int \tilde{\chi}|\partial^{2n+1}\phi_0|^2 + |\partial^{2n}\phi_{-1,0,1}|^2$$
$$\lesssim |a|\big(E_{2n}[F](t) + E_{2n}[F](0) + E_{2n+1}[\phi_0](t) + E_{2n+1}[\phi_0](0)\big)$$

Estimate V: ϕ_0 with T

Use T.

Difference between initial and final energy of ϕ_0 (of order $2n+1$) controlled by spacetime integral of ϕ_0 (of order $2n+1$) in suppχ.

Right?

WRONG!

Model problem

Consider u from $\mathbb{R} \times \mathbb{R} \times S^2 \ni (t, x, \omega)$ to $\mathbb{C}$

$$\left(\partial_t^2 - \partial_x^2 + \frac{1}{1+x^2}(-\Delta_{S^2} + 1000) + i\epsilon \underbrace{\frac{1}{1+x^{1000}}}_{W} \right) u = 0.$$

Noether's theorem does **not** provide a conserved energy (which is positive + small complex part).
Error term:

$$\epsilon W \operatorname{Im}(u \partial_t u).$$

Can't use A estimate and Cauchy-Schwarz because estimate on $|\partial_t u|^2$ fails near trapped region.

Replace $A = f(x)\partial_x$ by $f(|\partial_t|^b x)\partial_x$ to gain control of an extra $3b \leq 3/2$ derivatives of u.

Control of $u|\partial_t|u$ closes bootstrap.

This needs ∂_t pseudo-differential tricks.

- Convolution instead?
- Avoid Fackerel-Ipser?