Hidden symmetries and decay for the wave equation outside a Kerr black hole

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Hidden symmetries and Kerr wave decay

Joint with Lars Andersson.

- Kerr spacetime
  - parameters: M mass and Ma angular momentum.
  - rotating black hole, expected end state.
  - black hole for $|a| \leq M$; $a = 0$ is Schwarzschild.
- Wave: $\nabla^\alpha \nabla_\alpha \psi = 0$, decoupled, important equation, model.
- Goal: robust tools (hopefully) for Kerr stability.
- We consider $|a| \ll M$, exterior $r > r_+$.
- Result: $t^{-1+|a|C}$ decay for $|a| \ll M$ [arXiv:0908.2265].

Minkowski stability

- Friedrich
- Christodoulou- Klainerman
- Lindblad- Rodnianski

Kerr wave decay: other results

- $|a| \leq M$, mode decay: Finster-Kamran-Smoller-Yau
- $|a| \ll M$: Dafermos-Rodnianski, Tataru-Tohaneanu, Tataru
- (Builds on earlier Schwarzschild Morawetz and conformal energy results: Łaba- Soffer, B- Soffer, B- Sterbenz, Dafermos- Rodnianski, Metcalfe- Marzuola- Tataru- Tohaneanu, Luk. See also Donninger- Schlag- Soffer.)
- spectral and scattering results.

Energy momentum tensor

Energy-momentum tensor:

$$\begin{aligned} T[\psi]_{\alpha\beta} =& \nabla_\alpha\psi\nabla_\beta\psi - g_{\alpha\beta}(\nabla_\gamma\psi\nabla^\gamma\psi). \\ P_X[\psi]_\alpha =& T[\psi]_{\alpha\beta}X^\beta, \\ E_X[\psi](\Sigma) =& \int_\Sigma P_X[\psi]_\alpha \mathrm{d}\nu^\alpha. \end{aligned}$$

Properties:

1. $\mathbf{T}$ timelike, $\implies$ $E_\mathbf{T} \geq 0$.
2. S a symmetry, $\implies$ Same properties for $E_X[S\psi]$.
   (generalised symmetry S: $\nabla^\alpha\nabla_\alpha\psi = 0$ gives $\nabla^\alpha\nabla_\alpha S\psi = 0$)
3. $E_X(t_2) - E_X(t_1) = \int T[\psi]_{\alpha\beta}\nabla^{(\alpha}X^{\beta)}\mathrm{d}^4\mu_g$.

Schwarzschild basics

- Exterior region $(t, r, (\theta, \phi)) = (t, r, \omega) \in \mathbb{R} \times (2M, \infty) \times S^2$.
- Also use Regge-Wheeler coordinate $\frac{dr}{dr_*} = (1 - \frac{2M}{r})$, $r(0) = 3M$.
- Use $'$ for radial derivative.
- Rewrite the wave equation $\Box u$ for $\tilde{u} = ru$.

$$
\begin{aligned}
0 =& \tilde{u}'' + V\tilde{u} + \mathcal{R}\tilde{u} \\
\mathcal{R} =& \mathcal{R}_{\partial_t^2}\partial_t^2 + \mathcal{R}_{\Delta_{S^2}}\Delta_{S^2} \\
=& -\partial_t^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\Delta_{S^2}
\end{aligned}
$$

- Killing vectors: $\mathbb{S}_1 = \{\partial_t, \Theta_x, \Theta_y, \Theta_z\}$.

Energy

$\mathbf{T} = \partial_t$ is a timelike Killing vector
$\implies$ it generates a positive conserved energy.
Or, by integration by parts:

$$\begin{aligned}
0 =&(\partial_t \tilde{u})(\partial_{r_*}^2 \tilde{u} + V\tilde{u} + \mathcal{R}\tilde{u}) \\
=&(\partial_t \tilde{u})(-\partial_t^2 \tilde{u} + \partial_{r_*}^2 \tilde{u} + V\tilde{u} + \mathcal{R}_{\Delta_{S^2}} \Delta_{S^2}) \\
=&0 + \partial_t(\frac{1}{2}\left((\partial_t \tilde{u})^2 + (\partial_{r_*}\tilde{u})^2 + V\tilde{u}^2 + \mathcal{R}_{\Delta_{S^2}} | \not\nabla \tilde{u}|^2\right) \\
&+ \partial_{r_*}((r-2M)(\text{terms})) + \not\nabla \cdot (\text{terms}). \\
E_{\mathbf{T}}(t) =& \int_{\Sigma_t} (\partial_t \tilde{u})^2 + (\partial_{r_*}\tilde{u})^2 + V\tilde{u}^2 + \mathcal{R}_{\Delta_{S^2}} | \not\nabla \tilde{u}|^2 \mathrm{d}^3 x \\
E_{\mathbf{T}}(t_2) =& E_{\mathbf{T}}(t_1).
\end{aligned}$$

Higher energy and L^∞ estimate

Second-order symmetries:

$$\mathbb{S}_2 = \mathbb{S}_1^2 = \{\partial_t^2, \partial_t\Theta_i, \Theta_i\Theta_j\},$$

$$E_{\mathbf{T},n+1}[\tilde{u}](t) = \sum_{i=0}^{n-1} \sum_{S\in\mathbb{S}_n} E_{\mathbf{T}}[S\tilde{u}](t).$$

Heuristically, for $r > r_0 > 2M$:

$$\begin{aligned} < sup\Sigma_t, r > r_0 \|\tilde{u}\|^2_{L^\infty(\Sigma_t)} &\lesssim sup_{r>r_0} \|\Delta_{S^2} < tsolu\|^2_{L^2(S^2)} \\ &\lesssim \|\partial_{r_*}\Delta_{S^2}\tilde{u}\|^2_{L^2(\Sigma_t)} \\ &\lesssim E_{\mathbf{T}}[\Delta_{S^2}\tilde{u}](t) \\ &\lesssim E_{\mathbf{T},3}[\tilde{u}](t) \end{aligned}$$

Model black holes

[Figure goes here]

- Some null geodesics terminate on i^+.
- Assume $\mathcal{M} \sim \mathbb{R} \times (r_+, \infty) \times S^2$.
- In stationary spacetime, consider **null orbits** -null geodesics which, projected in the quotient, are constant or periodic.
- In Schwarzschild, null orbits at $r = 3M$. In Kerr, at $3M + O(a)$. All unstable.

Local energy decay

Goal:
For $r_+ < r_1 < r_2 < \infty$
and $(\mathbb{R} \times [r_1, r_2] \times S^2) \cap \{\text{null orbits}\} = \emptyset$.

$$\int_{\mathbb{R}} \int_{r_1}^{r_2} \int_{S^2} |\partial_t \tilde{u}|^2 + |\partial_{r_*} \tilde{u}|^2 + |\not\nabla \tilde{u}|^2 \mathrm{d}^4 x.$$

Morawetz (local energy) estimate in Schwarzschild

$$\mathbf{A} = \mathcal{F}\partial_{r_*}$$

(additional terms)
Illustrate method by integration by parts:

$$\begin{aligned} 0 =&(\mathcal{F}\partial_{r_*}\tilde{u})(\partial^2_{r_*}\tilde{u} + V\tilde{u} + \mathcal{R}\tilde{u}) \\ =&(\partial_{r_*}\tilde{u})\frac{1}{2}(\mathcal{F}')(\partial_{r_*}\tilde{u}) + \tilde{u}(-\mathcal{F})(\partial_{r_*}\mathcal{R})\tilde{u} \\ &+ \text{l.o.t.s} \\ &+ \partial_t(\mathcal{F}\tilde{u}'\partial_t\tilde{u}) + \partial_r((1-2M/r)(\text{terms})). \end{aligned}$$

$\mathcal{F}$ bounded $\implies \int_{\Sigma_t} |\mathcal{F}\tilde{u}'\partial_t\tilde{u}| \mathrm{d}^3\mu \leq E_{\mathbf{T}}$.

Integrate over spacetime:

$$CE_{\mathbf{T}} \geq I + II + \text{l.o.t.s},$$
$$I = \int_{\Sigma_t} (\partial_{r_*}\tilde{u})(\mathcal{F}')(\partial_{r_*}\tilde{u})\mathrm{d}^3\mu,$$
$$II = \int_{\Sigma_t} \tilde{u}(-\mathcal{F})\left(\partial_{r_*}\mathcal{R}\right)\tilde{u}\mathrm{d}^3\mu$$
$$(\partial_{r_*}\mathcal{R}) = \partial_{r_*}(-\partial_t^2 + \mathcal{R}_{\Delta_{S^2}}\Delta_{S^2}) = \partial_r\left(\frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right)\Delta_{S^2}.$$

We want I and II to be positive (or nonnegative). Idea:

- Choose $\partial_{r_*}\mathcal{R} \geq 0$
- Choose $(-\mathcal{F})(\partial_{r_*}\mathcal{R}) = (-\mathcal{F})\partial_r\left(\frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right)\Delta_{S^2}$ elliptic.

Schwarzschild potentials

[Sketch of $\mathcal{R}_{S^2}$]

- Take $\mathcal{F} = -w(\partial_{r_*}\mathcal{R}_{S^2})$, w positive.
  $\therefore (-\mathcal{F})(\partial_{r_*}\mathcal{R}_{\Delta_{S^2}})\Delta_{S^2} = w(\partial_{r_*}\mathcal{R}_{\Delta_{S^2}})^2\Delta_{S^2}$ is elliptic.
- Near $r = 3M$, convexity of $\mathcal{R}_{\Delta_{S^2}}$ implies $\mathcal{F}' > 0$.
  Away from $r = 3M$, choose w to get $\mathcal{F}' > 0$.

Decay rates

Use the vector

$$\mathbf{K} = (t^2 + r_*^2)\partial_t + 2tr_*\partial_{r_*}$$

to generate an energy which is $\sim t^2$ stronger than $\mathbf{T}$ energy.
The Morawetz (local energy) estimate (and tricks) gives a uniform bound on this energy, from which one concludes for $2M < r_1 < r < r_2 < \infty$, $u(t, r, \omega) \lesssim t^{-1}$.
The corresponding decay rate at null infinity follows by the same argument. Using transport equations and Stokes' theorem, the t^{-1} decay rate can be extended to the horizon.

Problems:

1. No timelike, Killing vector: no positive, conserved energy.
2. $\partial_t, \partial_\phi$ only Killing vectors:
   $E_{\mathbf{T}}[k^n u]$ doesn't control Sobolev norms,
3. Photon orbits: "trapping".
   Can't prove Morawetz estimate using a vector field.
4. (No timelike, Killing vector for **K**).

Energy

Use blended energy

- Stationary vector field timelike for r large ∂_t.
- Null generator extension timelike for r near r_+ $\partial + \omega_H \partial_\phi$.
- For $|a|$ small overlap.

Let

$$T_\chi = \partial_t + \chi \omega_H \partial_\phi .$$

Timelike in full exterior.
Failure to be conserved controlled by Morawetz (local decay) estimate.

Hidden symmetries

Hidden symmetry from Carter Killing 2-tensor

$$Q = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{\cos^2\theta}{\sin}\theta\partial_\phi^2 + a^2\sin^2\theta\partial_t^2.$$

Symmetry algebra

$$\mathbb{S}_2 = \{S_{\underline{a}}\}_{\underline{a}} = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, Q\},$$

$$|\Delta_{S^2}u|^2 \leq |Qu|^2 + |\partial_t^2 u|^2 + |\partial_\phi^2 u|^2,$$

$$E_{\mathbf{T}}[\Delta_{S^2}u] \leq \sum_{\underline{a}} E_{\mathbf{T}}[S_{\underline{a}}u].$$

Wave equation

Wave equation:

$$
\begin{aligned}
0 &= \partial_r \Delta \partial_r u + \frac{1}{\Delta}\mathcal{R}u \\
&= \partial_r \Delta \partial_r u + \frac{1}{\Delta}\mathcal{R}^{\underline{a}} S_{\underline{a}} u
\end{aligned}
$$

For Morawetz:

$$
\begin{aligned}
I &= \int_{\Sigma_t} (\partial_{r_*}\tilde{u})(\mathcal{F}')(\partial_{r_*}\tilde{u}) \mathrm{d}^3\mu \geq 0, \\
II &= \int_{\Sigma_t} \tilde{u}(-\mathcal{F})\left(\partial_{r_*}\mathcal{R}\right)\tilde{u}\mathrm{d}^3\mu \geq 0
\end{aligned}
$$

Idea:

1. $\mathcal{F} = -w(\partial_{r_*}\mathcal{R}^{\underline{a}})S_{\underline{a}}$ gives $w(\partial_{r_*}\mathcal{R})^2 \implies II \geq 0$.
2. Instability of null orbits $\implies \mathcal{F}'$ is elliptic, $\implies I \geq 0$.

A few technical details

Let

$$\mathcal{L} = \mathcal{L}^{\underline{a}} S_{\underline{a}} = \mathrm{Q} + \partial_t^2 + \partial_\phi^2,$$
$$\mathcal{F}^{\underline{a}} = -w(\partial_r \mathcal{R}^{\underline{a}})$$
$$\mathbf{A}^{\underline{ab}} = \mathcal{F}^{(\underline{a}} \mathcal{L}^{\underline{b})} \partial_r,$$
$$T[u]_{\underline{ab}\alpha\beta} = (1/4) \left(T[S_{\underline{a}} u + S_{\underline{b}} u]_{\alpha\beta} - T[S_{\underline{a}} u + S_{\underline{b}} u]_{\alpha\beta} \right),$$
$$E_{\mathbf{A}^{\underline{ab}}} = \int_{\Sigma_t} T_{\underline{ab}\alpha\beta} \mathbf{A}^{\underline{ab}\alpha} \mathrm{d}\nu^\beta$$

The argument then goes like a standard energy-momentum argument.
Boot strap with small a gives bounded energy and Morawetz (local energy) estimate.

Conformal energy

Let

$$\mathbf{K} = (t^2 + r_*^2) T_\chi + 2\tilde{N} t r_* \partial_{r_*},$$
$$\tilde{N} = \frac{(r^2 + a^2)^2}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}.$$

$\tilde{N}$ is used to cancel worst term in deformation.
Lower-order term for second worst.

Pointwise decay (stationary decay)

For r fixed, $\mathbf{K} \sim t^2 T_\chi$, so local energy decays like $t^{-2+|a|C}$.

$$|\psi(t,r,\theta,\phi)| \leq t^{-1+|a|C'} C \left(E_{\mathbf{K},3}(1) + E_{T_\chi,7}\right)$$

- Near $\mathcal{I}^+$, $(t+r_*)^{-1/2+C|a|}(t-r_*)^{-1}$,
- Near $\mathcal{H}^+$, $(t-r_*)^{-1+C|a|}$.

(Use hyperboloidal surfaces instead of null surfaces near $\mathcal{I}^+$.)

Maxwell

Take null frame L, N, e^A, e^B. Null decomposition:

$$\phi_0 = F(L,N) + iF(e^A - ie^B, e^A + ie^B) \qquad \rho = F(L,N)$$

$$F_{\text{span}(e^A,e^B)AB} = \sigma\epsilon_{AB}.$$

Wave-like equation: $\Box\phi_0 + a^2 W(r,\theta)\phi_0 = 0$. For small a, not ϕ symmetric, get similar decay.
For ϕ rotation symetric components need to project our stationary modes.
Use of Q subtle.

Why is there a loss?

For $E_{\mathbf{K},3}$, worst remaining term in deformation like

$$|a|t^2\mathbf{1}_{\mathrm{supp}\chi'}|\partial_r\psi|_2|\partial_\phi\psi|_2.$$

Morawetz estimate gives roughly a gain of t^{-1} for integrability.

$$\begin{aligned}
E_{\mathbf{K},3}(T) - E_{\mathbf{K},3}(1) \leq& |a|C\int_1^T t^{-1}E_{\mathbf{K},3}(t)\mathrm{d}t + \ldots,\\
\sim \frac{\mathrm{d}}{\mathrm{d}t}E_{\mathbf{K},3} =& |a|Ct^{-1}E_{\mathbf{K},3}\\
E_{\mathbf{K},3} \leq& t^{|a|C'}CE_{\mathbf{K},3}(1)\\
&+ CE_{T_\chi,7}.
\end{aligned}$$