Decay for the Maxwell field on the Schwarzschild manifold

Pieter Blue

14 January 2007

University of Edinburgh/ Maxwell Institute

Outline

- Example: Nonlinear wave decay
- Motivation: Black hole stability
- Model results: Maxwell decay

Wave equation

Function $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$,

$$\left(-\partial_t^2 + \sum_{j=1}^{3} \partial_{x_j}^2\right) u(t,x) = 0, \tag{1a}$$

$$u(0,x) = u_0(x), \tag{1b}$$

$$\partial_t u(0,x) = u_1(x). \tag{1c}$$

Energy

$$
\begin{aligned}
0 =& (\partial_t u)\left(-\partial_t^2 u + \Delta u\right) \\
=& -\partial_t \frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right) + \nabla \cdot \left(\partial_t u \nabla u\right),
\end{aligned}
$$

$$
\begin{aligned}
E_{\partial_t}[u](t) =& \frac{1}{2}\int_{\{t\}\times\mathbb{R}^3} |\partial_t u|^2 + |\nabla u|^2 d^3x \\
=& E_{\partial_t}[u](0).
\end{aligned}
$$

Sobolev estimates

- Sobolev estimate:

$$|f(\vec{x})| \leq \left(\int_{\mathbb{R}^3} |f|^2 + |\nabla^2 f|^2 d^3 x \right)^{1/2}.$$

- ∂_{x_j} is a symmetry of $\mathbb{R}^{1+3}$ $\Rightarrow$ if u solves (1a), then so does $\partial_{x_j} u$.
-

$$\sum_{j=1}^{3} E_{\partial_t}[\partial_{x_j} u](t) = \sum_{j=1}^{3} E_{\partial_t}[\partial_{x_j} u](0),$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3, d^3x)} \leq C \left(\sum_{j=1}^{3} E_{\partial_t}[\partial_{x_j} u](t) \right)^{1/2}.$$

Conformal energy

$$K = (t^2+r^2)\partial_t + 2tr\partial_r \qquad\qquad +\partial_t,$$

$$\begin{aligned} E_K[u](t) = \frac{1}{4}\int & (t+r)^2\left((\partial_t+\partial_{r_*})u\right)^2 \\ & + (t-r)^2\left((\partial_t-\partial_{r_*})u\right)^2 \\ & + 2(t^2+r^2)|\nabla_{S^2}u|^2 \qquad r^2 dr d^2\omega \\ & + E_{\partial_t}[u](t), \\ = E_K[u](0). \end{aligned}$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3,d^3x)} \leq t^{-1}C\left(\sum_{j=1}^{3} E_K[\partial_{x_j}u](0)\right)^{1/2}.$$

Nonlinear wave

Consider

$$(-\partial_t^2 + \Delta)u = |u|^{p-1}u$$

with $p > 3$.

$$\begin{aligned} E_K[\partial_{x_j}u](t) &= \int (K\partial_{x_j}u)(\partial_{x_j}|u|^{p-1}u)d^3x + E_K[\partial_{x_j}u](0), \\ &\leq C\int_0^t (1+s)^{-p+2}ds\cdot \\ &\quad \sup_{0\leq s\leq t}\left(E_K[u](s) + \sum_{j=1}^3 E_K[\partial_{x_j}u](s)\right)^{(p+1)/2} \\ &\quad + E_K[\partial_{x_j}u](0). \end{aligned}$$

- By continuity, if

$$\sum E_K[\partial_{x_j} u](0) \leq \epsilon,$$

  this norm will be small for all $t \geq 0$.
- Small data nonlinear solutions decay.

General relativity in 1 slide

- M: space-time manifold.
- g: Lorentz (-,+,+,+ signature) pseudometric.
  - time-like vector: $g(v,v) < 0$,
  - null vector: $g(v,v) = 0$,
  - space-like vector: $g(v,v) > 0$.
  - Curve length: $\int \sqrt{|g(\dot\gamma,\dot\gamma)|}ds$.
- (Vacuum) Einstein equation:

$$\mathrm{Ric}[g] = 0.$$

Schwarzschild manifold

$$ds^2 = (1 - 2M/r)(-dt^2 + (1 - 2M/r)^{-2} dr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

$$\begin{aligned} dr_* &= (1 - 2M/r)^{-1} dr, \\ r = 3M &\Longleftrightarrow r_* = 0. \end{aligned}$$

Symmetries $\mathbb{T} = \{T, \Theta_i\}$.

$r = 3M$ photon sphere, orbiting geodesics $\Rightarrow$ **trapping**.

Black hole stability

- Black hole: has an asymptotically flat region and future-directed null geodesics which cannot reach it.
- Kerr family: 2 parameter family. Schwarzschild is subfamily.
- **Conjecture: all black holes asymptotically approach a Kerr manifold.**

Literature

- Regge-Wheeler
- Price
- Teukolsky
- Whiting
- Christodoulou-Klainerman
- Lindblad-Rodnianski
- Bachelot, Nicolas, Hafner
- Finster-Kamran-Smoller-Yau
- Dafermos-Rodnianski
- Blue-Sterbenz, Blue-Soffer, Dafermos-Rodnianski

Maxwell equations

$$\begin{aligned} F_{[\alpha\beta]} &= 0 & F &\in \wedge^2 . \\ \nabla_{[\gamma} F_{\alpha\beta]} &= 0, & dF &= 0, & \text{(Maxwell.a)} \\ \nabla^{\alpha} F_{\alpha\beta} &= 0, & d * F &= 0. & \text{(Maxwell.b)} \end{aligned}$$

- Decoupled problem.
- Initial data

$$F_{\alpha\beta}(\{0\} \times \mathbb{R} \times S^2) = (F_0)_{\alpha\beta}$$

(with constraint).

Electric and magnetic decomposition

- If $\{T, X_1, X_2, X_3\}$ is an orthonormal basis with T time-like, then

$$\vec{E}_X = F(T, X),$$
$$\vec{B}_X = F(Y, Z),$$

with $\{X, Y, Z\}$ a cyclic permuation of $\{X_1, X_2, X_3\}$.

- Natural choice:

$$\hat{T} = (1 - 2M/r)^{-1/2} \partial_t,$$
$$\hat{R} = (1 - 2M/r)^{-1/2} \partial_{r_*},$$
$$\hat{\Theta} = r^{-1} \partial_\theta,$$
$$\hat{\Phi} = r^{-1} \sin^{-1} \theta \partial_\phi.$$

Difficulties

- K is not a symmetry.
- Trapping disrupts decay.
- Choice of components.
- System.
- Shortage of symmetries.

Spinor decomposition

$$\hat{l} = \hat{T} + \hat{R},$$
$$\hat{n} = \hat{T} - \hat{R},$$
$$m = \hat{\Theta} + i\hat{\Phi}.$$

$$\phi_1 = F(\hat{l}, m)$$
$$\phi_0 = \frac{1}{2}(F(\hat{l}, \hat{n}) + iF(m, \bar{m}))$$
$$\phi_{-1} = F(\hat{n}, \bar{m}),$$

$$u_+ = t + r_*,$$
$$u_- = t - r_*.$$

Vector fields: "conserved quantities"

Energy-momentum tensor:

$$\mathbf{T}_{\alpha\beta}.$$

Energy:

$$T = \partial_t,$$

$$E_T[F](t)$$
$$= \int_{\{t\}\times\mathbb{R}\times S^2} \mathbf{T}_{\alpha\beta} T^\alpha d\nu^\beta$$
$$= \int_{\{t\}\times\mathbb{R}\times S^2} \left(\frac{|\phi_1|^2}{4} + \frac{|\phi_0|^2}{2} + \frac{|\phi_{-1}|^2}{4} \right) (1 - 2M/r) r^2 dr_* d^2\omega,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_T[F](0) = 0.$$

$$K = (t^2 + {r_*}^2)\partial_t + 2tr_*\partial_{r_*}$$

$$E_K[F](t) =$$
$$\frac{1}{4}\int \left(u_+^2|\phi_1|^2 + (u_+^2 + u_-^2)|\phi_0|^2 + u_-^2|\phi_{-1}|^2\right)(1 - 2M/r)dr_* d^2\omega.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_K[F](t)$$
$$= \int_{\{t\}\times\mathbb{R}\times S^2} t\underbrace{\left(1 - \frac{r_*}{r}\left(1 - \frac{3M}{r}\right)\right)}_{\text{Positive in compact set}}|\phi_0|^2(1 - 2M/r)r^2 dr_* d^2\omega.$$

Price equations

$$
\begin{aligned}
N\Phi_1 &= M\Phi_0(1-2M/r)r^{-2},\\
L\Phi_0 &= \bar{M}\Phi_1 + \cot\theta\Phi_1,\\
N\Phi_0 &= -M\Phi_{-1} - \cot\theta\Phi_{-1},\\
L\Phi_{-1} &= -\bar{M}\Phi_0(1-2M/r)r^{-2}.
\end{aligned}
$$

$$
(-\partial_t^2 + \partial_{r_*}^2 + r^{-2}(1-2M/r)\Delta_{S^2})\Phi_0 = 0.
$$

Apply wave equation analysis (spin reduction).

$$
\sum_{k=0}^{n} E_K[\mathcal{L}_{\mathbb{T}}^k F](0) \leq C\left(\sum_{k=0}^{n+1} E_K[\mathcal{L}_{\mathbb{T}}^k F](0) + \sum_{k=0}^{n+5} E_T[\mathcal{L}_{\mathbb{T}}^k F](0)\right).
$$

Derivative trading

- Control derivatives in t and angular directions, but not radial direction.
- Define pointwise norms:

$$\mathbb{X} = \{T, R, \Theta, \Phi\},$$
$$|F|_{\mathbb{X},n,\mathbb{T}} = \sum_{k=0}^{n} \sum_{X_j \in \mathbb{X}} |\mathcal{L}_{\mathbb{T}}^{k} F(X_1, X_2)|.$$

- Locally derivatives of $\mathbb{X}$ with respect to $\mathbb{X}$ is multiple of $\mathbb{X}$.
- Maxwell equations relate radial derivatives to other derivatives:

$$\sum_{X_j \in \mathbb{X}} |\mathcal{L}_{\mathbb{T}}^{n} \mathcal{L}_R F(X_1, X_2)| \leq |F|_{\mathbb{X},n+1,\mathbb{T}}.$$

Main Result: Decay estimates

Theorem

If F is a solution of (Maxwell.a)-(Maxwell.b) *and* $2M < r_1 < r_2 < \infty$, *then there is a constant* $C_{(r_1,r_2)}$ *such that for all* $t \geq 0, r \in (r_1, r_2), (\theta, \phi) \in S^2$,

$$|\vec{E}(t,r,\theta,\phi)| + |\vec{B}(t,r,\theta,\phi)| \leq Ct^{-1}\left(\sum_{k=0}^{4} E_K[\mathcal{L}_{\mathbb{T}}^k F](0) + \sum_{k=0}^{8} E_T[\mathcal{L}_{\mathbb{T}}^k F](0)\right).$$

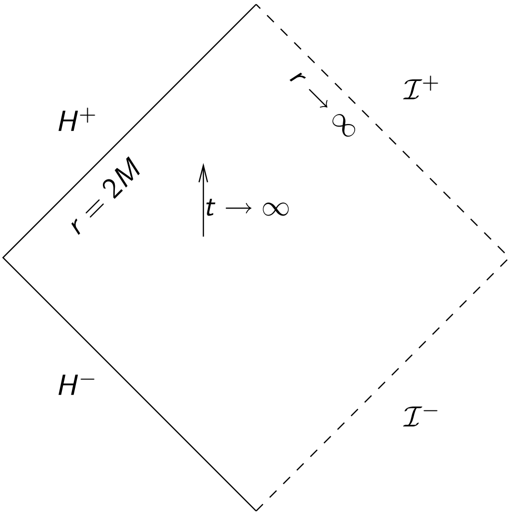
(Also, decay at null infinity and at the event horizon.)

Future work

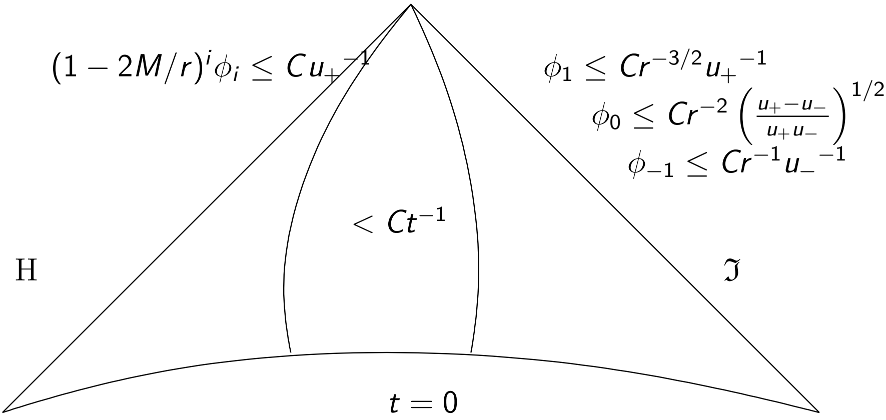
- Reach conjectured decay
- Yang-Mills and linearized gravity on Schwarzschild
- Kerr

Conformal diagram

$$u_- = t - r_*, \qquad u_+ = t + r_*. \qquad U_\pm = \arctan u_\pm.$$



Main result: Decay estimates



Constant depends on energies of F and 8 derivatives. Initial data need not be restricted to finitely many spherical harmonics or to vanish on the event horizon.