

Decay for fields outside black holes

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- ▶ Schwarzschild and Kerr are explicit black holes.
- ▶ The wave equation is a model for black hole stability.
- ▶ Scattering, explicit representation and L^∞ decay.
- ▶ Vector-field methods and decay on Schwarzschild.
- ▶ (Captures two types of wave decay.)
- ▶ (Maxwell field works very similarly.)
- ▶ Uniform boundedness on Kerr.
- ▶ Work in progress with Lars Andersson.

Previous work

- ▶ Price
- ▶ Wald, Kay-Wald
- ▶ Bachelot, Nicolas, Häfner
- ▶ Finster-Kamran-Smoller-Yau
- ▶ Łaba-Soffer
- ▶ Blue-Soffer
- ▶ Blue-Sterbenz
- ▶ Dafermos-Rodnianski
- ▶ Marzoula-Metcalf-Tataru-Tohaneanu

Conventions

- ▶ \mathcal{M}^{1+3} is the space-time manifold.
- ▶ g : Lorentz $(-, +, +, +)$ signature) pseudometric.
- ▶ The pseudometric g defines a covariant derivative ∇_α
- ▶ The wave equation is

$$\square u = g^{\alpha\beta} \nabla_\alpha \nabla_\beta u = 0.$$

- ▶ We will assume $\mathcal{M}^{1+3} = \mathbb{R} \times M^3$.
- ▶ We will assume that each hypersurface $\{t\} \times M^3$ has a common (conformal) boundary, that this is a foliation by Cauchy surfaces, etc.

Vector field method (Part I)

- ▶ γ a null geodesic:
 - ▶ energy associated to X : $e_X[\gamma](t) = g_{\alpha\beta}\dot{\gamma}(t)^\alpha X^\beta$.
 - ▶ T time-like: $e_T[\gamma](t) \geq 0$.
 - ▶ X Killing ($\nabla^\alpha X^\beta = -\nabla^\beta X^\alpha$): $e_X[\gamma](t_2) = e_X[\gamma](t_1)$.
- ▶ u solving the wave equation:
 - ▶ Energy-momentum(-stress) tensor and energy associated to X :

$$\mathbf{T}_{\alpha\beta} = (\nabla_\alpha u)(\nabla_\beta u) - \frac{1}{2}g_{\alpha\beta}(\nabla^\gamma u)(\nabla_\gamma u),$$

$$E_X[u](t) = \int_{\{t\} \times M} \mathbf{T}_{\alpha\beta} X^\alpha d\nu^\beta.$$

- ▶ T time-like: $E_T[u](t) \geq 0$.
- ▶ X Killing: $E_X[u](t_2) = E_X[u](t_1)$.
- ▶ (For a function q , $E_q = \int (qu\nabla_\alpha u - (1/2)(\nabla_\alpha q)u^2)d\nu^\alpha$.)

If X is not Killing

$$E_X[u](t_2) - E_X[u](t_1) = \int_{t_1}^{t_2} \int_M \nabla^\alpha X^\beta \mathbf{T}_{\alpha\beta} d^4x$$

Vector field method (Part II)

- ▶ If Y is a Killing vector, and $\square u = 0$, then $\square(Yu) = 0$.
- ▶ If $\mathbb{S}_1 = \{Y_1, \dots, Y_k\}$ is a set of Killing vectors (which is closed under commutators), and T is a time-like, Killing vector, then let

$$E_{T,1}[u](t) = E_T[u](t),$$

$$E_{T,2}[u](t) = \sum_{Y \in \mathbb{S}_1} E_T[Yu](t),$$

$$E_{T,n+1}[u](t) = \sum_{k=1}^n \sum_{Y_1, \dots, Y_k \in \mathbb{S}_1} E_T[Y_k \dots Y_1 u](t).$$

- ▶ These are also conserved.

- ▶ Cartesian coordinates $x^0 = t, x^i$.
- ▶ $g_{00} = -1, g_{11} = g_{22} = g_{33} = 1$, off-diagonal are zero.
- ▶ For $u(t, \vec{x})$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta u = \left(-\partial_t^2 + \sum \partial_{x_i}^2 \right) u = 0.$$

- ▶ $T = \partial_t$ is Killing, time-like.

$$E_{T,1}[u](t) = E_T[u] = \int |\partial_t u|^2 + \sum |\partial_{x_i} u|^2 d^3x,$$
$$E_T[u](t_2) = E_T[u](t_1).$$

- ▶ $\mathbb{S}_1 = \{\partial_t, \partial_{x_i}\}$ is a set of Killing vectors.

$$E_{T,2} = \sum_{Y \in \mathbb{S}_1} E_T[Yu] = \int |\partial_t^2 u|^2 + \sum |\partial_t \partial_{x_i} u|^2 + \sum |\partial_{x_i} \partial_{x_j} u|^2 d^3x.$$

- ▶ Sobolev estimate:

$$|v(\vec{x})|^2 \lesssim \int_{\mathbb{R}^3} |\partial_{x_i} \partial_{x_j} v|^2 d^3x.$$

- ▶ (Need only 3/2 derivatives.)
- ▶ For all (t, \vec{x}) :

$$|u(t, \vec{x})|^2 \lesssim E_{T,2}(t_1).$$

- ▶ $K = (t^2 + r^2)\partial_t + 2tr\partial_r$ is (conformal) Killing.

$$E_K[u](t) = \int_{\{t\} \times \mathbb{R}^+ \times S^2} (t \pm r)^2 |(\partial_t \pm \partial_r)u|^2 + (t^2 + r^2) |\nabla_{S^2} u|^2 d^2\omega$$

$$E_K[u](t_2) = E_K[u](t_1).$$

- ▶ asymptotically, energy carried by in- and out-going pieces.
- ▶ Local energy $\lesssim t^{-2} E_K[u](t_1)$.
- ▶ $|u(t, \vec{x})| \lesssim t^{-1} E_{K,2}(t_1)^{1/2}$.

Geometry of Schwarzschild

- ▶ Mass M .
- ▶ Spherical coordinates (t, r, θ, ϕ) :

$$g = - (1 - 2M/r) dt^2 + (1 - 2M/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

- ▶ Exterior $r > 2M$, $t \in \mathbb{R}$, $(\theta, \phi) \in S^2$.
- ▶ $T = \partial_t$ is time-like, Killing.
- ▶ Θ_i are Killing.

- ▶ Three (independent) conserved quantities: e_T , e_{Θ_i} ; and null-geodesic condition $g(\dot{\gamma}, \dot{\gamma}) = 0$. As many equations as unknowns.

$$-\left(\frac{1}{r^4} \frac{d}{d\lambda} r\right)^2 = -e_T^2 + V_L \sum_i e_{\Theta_i}^2,$$
$$V_L = \frac{1}{r^2} (1 - 2M/r).$$

- ▶ Photon sphere: Orbiting geodesics at maximum of V_L , at $r = 3M$.

Problems of trapped geodesics

If γ is a geodesic in some manifold, and u is a solution to the wave equation on the manifold, then an arbitrary amount of the initial energy can remain arbitrarily close to the geodesic for an arbitrarily long period of time. [Ralston].

$$\forall \epsilon_1 > 0, \epsilon_2 > 0, T > 0 : \exists u : \forall t \leq T$$

$$\int_{\{t\} \times (\gamma + B(\epsilon_1, 0))} |\partial_t u|^2 + |\nabla u|^2 d^3x > (1 - \epsilon_2) E[u].$$

Schwarzschild wave equation

- ▶ New radial co-ordinate: $\frac{dr}{dr_*} = (1 - 2M/r)$.
- ▶ Wave equation for $\tilde{u} = ru$:

$$0 = -\partial_t^2 \tilde{u} + \partial_{r_*}^2 \tilde{u} + V_L \Delta_{S^2} \tilde{u} + V \tilde{u}, \quad (\text{Wave})$$

$$V = -2Mr^{-3} (1 - 2M/r).$$

- ▶ $T = \partial_t$ is time-like and Killing, so E_T gives a positive, conserved energy

$$E_T = \int |\partial_t \tilde{u}|^2 + |\partial_{r_*} \tilde{u}|^2 + V_L |\nabla_{S^2} \tilde{u}|^2 + V |\tilde{u}|^2 dr_* d^2\omega.$$

Decay on Schwarzschild

- ▶ “Conformal vector field” $K = (t^2 + r_*^2)\partial_t + 2tr_*\partial_{r_*}$,

$$E_K[u] = \int (t \pm r_*)^2 |(\partial_t \pm \partial_{r_*})u|^2 + (t^2 + r_*^2)(V_L |\nabla_{S^2} u|^2 + V |u|^2) dr_*$$

$$\frac{d}{dt} E_K \leq Ct \int_{r=3M-c}^{r=3M+c} |\nabla_{S^2} u|^2 + |u|^2 dr_* d^2\omega.$$

- ▶ Control over the integral of the angular derivatives (and the function) would give control over E_K , which would then give decay for the local energy.
- ▶ Applying angular derivatives to control Sobolev norms would give

$$|u(t, r, \theta, \phi)| \leq Ct^{-1} \left(\sum_i E_K[\Theta_i^2 u] \right)^{1/2}.$$

Morawetz estimate in Schwarzschild

- ▶ Introduce the Morawetz vector field

$$A = g(r_*)\partial_{r_*} = g(r_*(r))(1 - 2M/r)\partial_r.$$

(There are some lower-order correction terms)



$$\begin{aligned}\nabla^\alpha A^\beta \mathbf{T}_{\alpha\beta} &= (\partial_{r_*} g)(\partial_{r_*} u)^2 - gV'_L |\nabla_{S^2} u|^2 \\ &\quad + (\text{lower-order coefficients})|u|^2.\end{aligned}$$

- ▶ $g' \geq 0$ and $-gV'_L \geq 0$ give positive top-order terms.
- ▶ (Lower-order terms controlled by repulsivity and Hardy.
NEW: This can be done without harmonic decomposition.)
- ▶ g bounded implies $E_A \leq CE_T$.

- ▶ Choose g to go from negative to positive at photon sphere (maximum of V_L) so $-gV'_L \geq 0$, and increasing so $g' \geq 0$.

$$\begin{aligned}
 & \int_{t_1}^{t_2} (1 + r_*^2)^{-1} (\partial_r u)^2 \\
 & + (1 - 2M/r) \frac{(r - 3M)^2}{r^6} |\nabla_{S^2}|^2 \\
 & + (1 + r_*^2)^{-2} |u|^2 d^4x \leq E_A(t_2) - E_A(t_1) \\
 & \leq C(E_T(t_2) + E_T(t_1)) \\
 & \leq CE_T.
 \end{aligned}$$

- ▶ (with a boot strap)

$$E_K[u](t_2) \leq C \left(E_K[u](t_1) + \sum_i E_T[\Theta_i u](t_1) \right).$$

Summary of Schwarzschild

- ▶ T time-like, Killing gives positive, conserved energy.
- ▶ $A = g\partial_{r_*}$ (directed away from photon sphere) gives Morawetz (space-time integral bounds for square of derivatives).
- ▶ K gives positive energy. Get a uniform bound on growth of the associated energy using the Morawetz estimate.
- ▶ Get pointwise decay t^{-1} (for $r \geq r_0 > 2M$) from bound on $\sum_i E_K[\Theta_i^2 u]$.
- ▶ (Y gives better control of the energy at the event horizon. Similar decay rate for $|u|$ along horizon can be found.)

Geometry of Schwarzschild and Kerr

- ▶ Mass M , rotational parameter a .
- ▶ Schwarzschild is $a = 0$, subcritical Kerr is $|a| < M$.
- ▶ Spherical co-ordinates, (t, r, θ, ϕ) :

$$g = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mr a \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2$$
$$+ \Sigma d\theta^2 + ((r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta) \frac{\sin^2 \theta}{\Sigma} d\phi^2,$$
$$\Sigma = r^2 + a^2 \cos^2 \theta,$$
$$\Delta = r^2 - 2Mr + a^2.$$

- ▶ Exterior: $r > r_+ = M + \sqrt{M^2 - a^2}$.
- ▶ Symmetries: $\partial_t, \partial_\phi$.

Null geodesics in Kerr

- ▶ Only two conserved quantities from classical symmetries: e_{∂_t} and $e_{\partial_\phi} = e_{L_z}$; one more from null-geodesic condition: $g(\dot{\gamma}, \dot{\gamma}) = 0$. Still “obvious conserved quantities” are fewer than number of equations.
- ▶ There’s a “hidden symmetry”:

$$Q = e_{\Theta_x}^2 + e_{\Theta_y}^2 + a^2 \sin^2 \theta e_{\partial_t}^2.$$

$$\left(\frac{1}{\Sigma(r^2 + a^2)} \frac{dr}{d\lambda} \right)^2 = \tilde{\mathcal{R}}$$

$$\begin{aligned} \tilde{\mathcal{R}}(r; M, a; e_{\partial_t}, e_{\partial_\phi}, Q) = & -e_{\partial_t}^2 - \frac{4aMr}{(r^2 + a^2)^2} e_{\partial_t} e_{L_z} \\ & + \frac{\Delta}{(r^2 + a^2)^2} Q + \frac{\Delta - a^2}{(r^2 + a^2)^2} e_{L_z}^2. \end{aligned}$$

Problems in Kerr

- ▶ There are photon spheres/ orbiting geodesics at different radii. This makes building a Morawetz estimate hard.
- ▶ $T_\infty = \partial_t$ is Killing, but it is not time-like near $r = r_+$.
- ▶ $(T_{r_+} = \partial_t + (a/(r_+^2 + a^2))\partial_\phi)$ is Killing near $r = r_+$ but not near $r \rightarrow \infty$.)
- ▶ There are not enough symmetries (in the angular directions) to generate higher norms with enough derivatives to make an elliptic estimate to control local L^∞ .

Solutions in Kerr

- ▶ There's an additional (almost) elliptic second-order operator that can be used to define higher norms

$$\mathcal{Q} = \Theta_x^2 + \Theta_y^2 + a^2 \sin^2 \theta \partial_t^2.$$

- ▶ For a small, we can use T_∞ near $r \rightarrow \infty$ and T_{r_+} near $r = r_+$, and “blend” the two vector fields together at some large distance R from the black hole. The resulting vector will be time-like, and it will only fail to be Killing in $r \in [R, R + M]$.
- ▶ If we had a Morawetz estimate, we could then control this failure to be Killing in a boot-strap argument.
- ▶ Wave equation is:

$$0 = \partial_r \Delta \partial_r u + \tilde{\mathcal{R}}(r; M, a; \partial_t, \partial_\phi, \mathcal{Q}).$$

- ▶ In the Morawetz vector-field need g to change-sign at photon spheres,
- ▶ Take $g \sim \partial_r \tilde{\mathcal{R}}$.

Hidden symmetries in part II of the vector-field method

- ▶ Let \mathbb{S}_2 be the set of second of second-order (possibly hidden) symmetries

$$\mathbb{S}_2 = \{\underline{S}_m\} = \{\partial_t^2, \partial_t \partial_\phi, \partial_\phi^2, Q\}.$$

- ▶ For a time-like vector, T , these can be used to define higher norms:

$$E_{T,3}[u](t) = \sum_{S \in \mathbb{S}_2} E_T[Su]$$

- ▶ The symmetry index allows expansions:

$$\begin{aligned} Q &= Q^m \underline{S}_m, \\ \mathcal{R} &= \mathcal{R}^m \underline{S}_m. \end{aligned}$$

(Hidden) symmetries in part I of the vector-field method

- ▶ Let

$$\mathbf{T}[u, v]_{\alpha\beta} = \frac{1}{4} (\mathbf{T}[u + v]_{\alpha\beta} - \mathbf{T}[u - v]_{\alpha\beta}).$$

- ▶ For a symmetric, double-indexed set of vector-fields $\{X^{\underline{ms}\alpha}\}$, define the associated energy to be

$$E_{X^{\underline{ms}}} = \int \mathbf{T}[S_{\underline{m}}u, S_{\underline{s}}u]_{\alpha\beta} X^{\underline{ms}\beta} d\nu^\alpha.$$

- ▶ This allows

$$A^{\underline{ms}} = g^{\underline{ms}} \partial_r,$$
$$g^{\underline{ms}} = (\mathcal{Q} + \partial_\phi^2 + \partial_t^2)^{(m} \otimes \partial_r \tilde{\mathcal{R}}^{\underline{s})}$$

Take

$$E_3 = \sum_{Y_i \in \{\partial_t, \partial_\phi\}} E_T[Y_2 Y_1 u] + E_T[Qu]$$

$$E_A(t) \leq CE_3(t)$$

$$\begin{aligned} E_3(t_2) - E_3(t_1) &\leq a \int_{t_1}^{t_2} \int_R^{R+M} \int_{S^2} (\text{third-derivatives of } u)^2 d^4x \\ &\leq aC(E_A(t_2) - E_A(t_1)) \\ &\leq aC(E_3(t_2) - E_3(t_1)), \end{aligned}$$

$$E_3(t_2) \leq \frac{1 + aC}{1 - aC} E_3(t_1).$$

- ▶ Let $K = (t^2 + r_*^2)(\partial_t + \omega_K \partial_\phi) + 2tr_*$.



$$\begin{aligned} \frac{d}{dt} E_K &\leq \int t(\text{Schwarzschild-like derivatives}) d^3x \\ &\quad + a \int t^2(\text{Other derivatives}) d^3x. \end{aligned}$$

- ▶ For sufficiently small a ,

$$E_K(t_2) \lesssim (E_K(t_1) + E_{T,3}(t_1))(t_2 - t_1)^{Ca},$$

- ▶ Local energy $\lesssim t^{-2+Ca} \|u\|(t_1)$.
- ▶ $|u(t, r_*, \theta, \phi)| \lesssim t^{-1+Ca} \|u\|(t_1)$.

- ▶ Close relation between null geodesic equation and wave equation.
- ▶ On Kerr:
 - ▶ No positive, conserved energy.
 - ▶ New Morawetz estimate gives alternate proof of uniformly bounded energy. (This uses no Fourier analysis in time and very little in space.)
 - ▶ New elliptic operator \mathcal{Q} quickly gives L^∞ bound from energy bound for $r \geq r_0 > r_+$.
 - ▶ K gives almost the same decay for $u \lesssim t^{-1+C_a} \|u\| (t_1)$.
 - ▶ Our choice of foliation does not immediately reveal decay at event horizon, which is clear in [Dafermos-Rodnianski].