Decay of waves on a warp product manifold with an unstable, closed geodesic surface.

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Dispersion on manifolds

For Schrödinger:

- $\mathbb{R}^n$: Strichartz.
- $\mathbb{T}^n$: ϵ loss, local in time [Bourgain]
- Compact manifolds: same [Burq-Gérard-Tzvetkov]
- Non-trapping (no closed geodesic): Strichartz [Hassel-Tao-Wunsch].

The problem with trapping

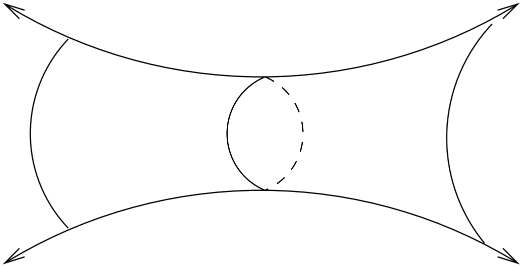
For wave:
An arbitrarily large amount of the energy can stay within a neighbourhood of a closed geodesic for an arbitrarily long time. [Ralston]

$$\forall T, \epsilon_1, \epsilon_2 : \exists \psi : \forall t \in [0, T] :$$
$$\int_{\text{geodesic}+B(0,\epsilon_1)} (|\dot{\psi}(t)|^2 + |\nabla \psi(t)|^2) > (1-\epsilon_2) E[\psi, \dot{\psi}](0)$$

Model problem

- Spherically-symmetric, single closed geodesic sphere

$$ds^2 = dr_*^2 + r(r_*)^2 d^2\omega$$



- $\Box_g \tilde{\phi} = 0$, let $\phi = r\tilde{\phi}$

$$-\partial_t^2 \phi = -\partial_{r_*}^2 \phi + V\phi + V_L(-\Delta_{S^2})\phi$$

Results

- If r has a single minimum, other assumptions

$$\|\tilde{\phi}\|_{\mathcal{L}^4_{tx}} < C$$

  C depends on weighted $H^{1+\epsilon}$ norm.
- Same for small data, defocusing $\Box_g \tilde{\phi} = |\phi|^{p-1}\phi$, $8/3 < p < 3$.
- (with A. Soffer)

Conformal Estimate

Energy and conformal charge

$$E[\phi,\dot\phi] = \frac{1}{2}\int |\dot\phi|^2 + |\partial_{r_*}\phi|^2 + V|\phi|^2 + V_L|\nabla_{S^2}\phi|^2 dr_* d^2\omega$$

$$E_{\mathcal{C}}[\phi,\dot\phi] = \frac{1}{4}\int |(t-r_*)(\dot\phi - \partial_{r_*}\phi)|^2 + |(t+r_*)(\dot\phi + \partial_{r_*}\phi)|^2 + 2(t^2+r_*^2)(V|\phi|^2 + V_L|\nabla_{S^2}\phi|^2) dr_* d^2\omega$$

Trapping

$$\frac{d}{dt}E_{\mathcal{C}} \lesssim t\int \chi |L\phi|^2 dr_* d^2\omega$$

χ compact, $L = \langle \nabla_{S^2} \rangle$

Morawetz estimate

- Spherical harmonic decomposition, $V_l = V + V_L(-\Delta_{S^2})$
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$$\gamma = g\partial_{r_*} + g'/2, \qquad g = \int_0^{r_* - (V_l \text{ peak})} (1+|x|)^{-2} dx$$

- Smoothed Morawetz-type estimate [Lavine]

$$\int\int \frac{|\phi|^2}{(1+|x|)^4} + \frac{|\partial_{r_*}\phi|^2}{(1+|x|)^2} - gV_L'|L\phi|^2 dr_* d^2\omega dt \lesssim E$$

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$$E_{\mathcal{C}}(t) \lesssim E_{\mathcal{C}}(0) + tE[L\phi, L\dot{\phi}](0)$$

Angular Modulation

$$\int\int \frac{|\phi|^2}{(1+|x|)^4} + \frac{|\partial_{r_*}\phi|^2}{(1+|x|)^2} - gV_L'|L\phi|^2 dr_* d^2\omega dt \lesssim E$$

- Goal: control more L derivatives
- Large r_*: Use $gV_L'|L\phi|^2$
- Small r_*: Rescale

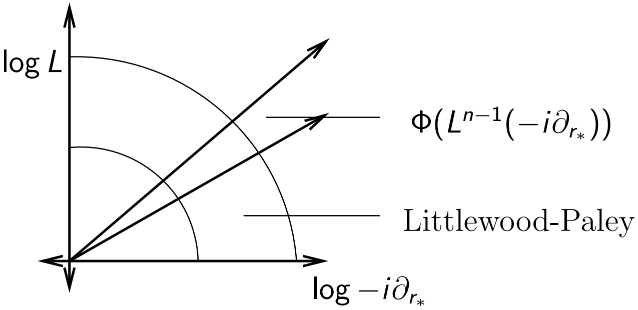
$$g_{L^m}(r_*) = g(L^m r_*)$$

- Balance: $m = 3/4$

$$\int\int \chi|L^{3/4}\phi|^2 dr_* d^2\omega dt \lesssim E$$

Phase Space Analysis

- Use $\chi|\partial_{r_*}\phi|^2$ control
- Frequency localise: $-i\partial_{r_*} \sim L^{1-n}$



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$$\int\int \chi|L^{1-\epsilon}\phi| dr_* d^2\omega dt \lesssim E$$

Temporal Refinement

- Multiply estimate by t,
- Localise inside light-cone $|r_*| < t$ (earlier work with J. Sterbenz).
- Control localised $t\dot{\phi}$ by conformal charge

$$\int t \int \chi |L^{1-\epsilon}\phi| dr_* d^2\omega dt \lesssim E_{\mathcal{C}}^{1/2} E^{1/2}.$$

Closing argument

- conformal-Sobolev

$$\|V_L^{1/3}\phi\|_{\mathcal{L}^6} \lesssim E_{\mathcal{C}}(t)t^{-2/3}$$
$$\|\phi\|_{\mathcal{L}^2} \lesssim E_{\mathcal{C}}(t)$$

- Persistance of regularity: weighted space-time integral norm
- Small data estimate for original solution

$$\|\tilde{\phi}\|^2_{\mathcal{L}^4(\mathcal{L}^4(r^2 dr_* d^2\omega),dt)} \lesssim E_{\mathcal{C}}(0) + E[L^\epsilon\phi, L^\epsilon\dot{\phi}](0)$$

NLS?

- Morawetz estimates: OK
- Conformal $\mapsto$ pseudoconformal: Probably