Decay for Maxwell's equation outside a Schwarzschild black-hole.

Pieter Blue

18 May 2007

Obstructions to decay

Photons orbit at $r = 3M$.

Waves on Riemannian manifold [Ralston]:

$$\forall T, \epsilon_1, \epsilon_2 > 0 : \exists \psi : \forall t \in [0, T] :$$

$$\int_{\text{geodesic}+B(0,\epsilon_1)} \text{energy density} > (1 - \epsilon_2)(\text{energy at } t = 0)$$

Relativistic Fields

- Price 1972: Schwarzschild wave, Maxwell, linearised gravity, t^{-2-2l}. Separability, spherical harmonic decomposition.
- Kay-Wald 1980's.
- Christodoulou-Klainerman 1990, 1992.
- Finster-Kamran-Smoller-Yau 1999-2007.
- Blue-Soffer 2004-2006.
- Blue-Sterbenz 2005.
- Dafermos-Rodnianski 2005.

Problem

Exterior $r > 0$.

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2).$$

Regge-Wheeler coord: $dr/dr_* = (1 - 2M/r)$, $r_* \in \mathbb{R}$.

Maxwell field specified on $t = 0$:

$$F_{ab} = -F_{ba}, \qquad \nabla^a F_{ab} = 0, \qquad \nabla_{[a}F_{bc]} = 0.$$

Components

Orthonormal and null tetrads

$$\hat{T}, \qquad \hat{R}, \qquad \hat{\Theta}, \qquad \hat{\Phi}.$$
$$l = \hat{T} + \hat{R}, \qquad n = \hat{T} - \hat{R}, \qquad m = \hat{\Theta} + i\hat{\Phi}, \qquad \bar{m}.$$

Maxwell components

$$\phi_1 = F(l, m), \quad \phi_0 = \frac{1}{2}(F(l, n) - F(m, \bar{m})), \quad \phi_{-1} = F(n, \bar{m}).$$

Energy

$$E_{\partial_t}[F] = \int_{\Sigma_t} (|\phi_1|^2 + |\phi_0|^2 + |\phi_{-1}|^2) r^2 dr d^2\omega.$$

Energy-Momentum integrals

Conformal energy

$$K = (t^2+{r_*}^2)\partial_t + 2tr_*\partial_{r_*}$$

$$E_K[F] = \int_{\Sigma_t}\Big((t+r_*)^2|\phi_1|^2 + (t^2+{r_*}^2)|\phi_0|^2 + (t-r_*)^2|\phi_{-1}|^2\Big)r^2drd^2\omega.$$

Trapping

$$\frac{d}{dt}E_K[F] \lesssim \int_{\Sigma_t} t\chi|\phi_0|^2r^2drd^2\omega.$$

χ supported near $r = 3M$.

Reduction to wave equation

Price equations (ignoring r dependent factors):

$$(\partial_t \pm \partial_{r_*})(\phi_j) \sim (\hat{\Theta} \mp i\hat{\Phi})\phi_{j\pm 1}$$

For $u = r^2\phi_0$,

$$(\partial_t^2 - \partial_{r_*}^2)u = \frac{1}{r^2}(1 - 2M/r)\Delta_{S^2} u,$$

$$\int\int t\chi |u|^2 dr_* d^2\omega dt \leq C(\tilde{E}_{\partial_t}[\nabla_{S^2}^2 u(0)] + \tilde{E}_K[u(0)]).$$

Decay in mean

Hypersurface energy flux $\leq \min(t + r_*, t - r_*)^{-2}$

$\implies$ decay in mean.

Sobolev estimates and commuting vector fields

Sobolev estimates:

$$|\phi(r_*, \omega)|^2 \lesssim \int_{\Sigma_t} |\nabla_{S^2}^2 \partial_{r_*} \phi|^2 + |\nabla_{S^2}^3 \phi|^2 dr_* d^2\omega.$$

If V is Killing, $\mathcal{L}_V F$ also solves Maxwells equation.

Schwarzscild: t and angular derivatives decay in mean.

Use Maxwell's equations to “trade” derivatives.

Decay

For $2M < R_1 < r < R_2$, $\exists C_{R_1,R_2}$:

$$|\phi_i| \leq t^{-1} C_{R_1,R_2} \Big(\sum_{j_0+j_1+j_2+j_3=0}^{5} \tilde{E}_T[\partial_t^{j_0} \partial_{\theta_1}^{j_1} \partial_{\theta_2}^{j_2} \partial_{\theta_3}^{j_3} u(0)] + \sum_{j_0+j_1+j_2+j_3=0}^{3} \tilde{E}_K[\partial_t^{j_0} \partial_{\theta_1}^{j_1} \partial_{\theta_2}^{j_2} \partial_{\theta_3}^{j_3} u(0)] + \sum_{j_0+j_1+j_2+j_3=0}^{3} E_K[\partial_t^{j_0} \partial_{\theta_1}^{j_1} \partial_{\theta_2}^{j_2} \partial_{\theta_3}^{j_3} F(0)] \Big)^{\frac{1}{2}}$$