Decay for the wave equation outside a slowly rotating Kerr black hole

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Hidden symmetries and Kerr wave decay

Joint with Lars Andersson.

- Kerr spacetime
  - parameters: M mass and Ma angular momentum.
  - rotating black hole, expected end state.
  - black hole for $|a| \leq M$; $a = 0$ is Schwarzschild.
- Wave: $\nabla^\alpha \nabla_\alpha \psi = 0$, decoupled, important equation, model.
- Goal: robust tools (hopefully) for Kerr stability.
- We consider $|a| \ll M$, exterior $r > r_+$.
- Result: $t^{-1+|a|C}$ decay for $|a| \ll M$ [arXiv:0908.2265].

General relativity in 1 slide

- $\mathcal{M}$: space-time manifold.
- g: Lorentz (-,+,+,+ signature) pseudometric.
  - Time-like vector: $g(v, v) < 0$,
  - null vector: $g(v, v) = 0$,
  - space-like vector: $g(v, v) > 0$.
  - Summation convention $g(v, v) = g_{\alpha\beta} v^\alpha v^\beta$.
  - Curve length: $\int \sqrt{|g(\dot\gamma, \dot\gamma)|} ds$.
- (Vacuum) Einstein equation:

$$\mathrm{Ric}[g] = 0.$$

Minkowski stability

Minkowski $\mathbb{R}^{3+1}, -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2$.

- Friedrich
- Christodoulou- Klainerman
- Lindblad- Rodnianski

Kerr wave decay: other results

- $|a| \leq M$, mode decay: Finster-Kamran-Smoller-Yau
- $|a| \ll M$: Dafermos-Rodnianski, Tataru-Tohaneanu, Tataru
- (Builds on earlier Schwarzschild Morawetz and conformal energy results: Łaba- Soffer, B- Soffer, B- Sterbenz, Dafermos- Rodnianski, Metcalfe- Marzuola- Tataru- Tohaneanu, Luk. See also Donninger- Schlag- Soffer.)
- spectral and scattering results.

Wave on Lorentz manifold

- The pseudometric g defines a covariant derivative ∇_α
- (and $g^{\alpha\beta}$ by $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha{}_\beta$ and $\nabla^\alpha = g^{\alpha\beta}\nabla_\beta$).
- The wave equation is

$$\Box u = g^{\alpha\beta}\nabla_\alpha\nabla_\beta u = 0.$$

- We will assume $\mathcal{M}^{1+3} = \mathbb{R} \times M^3$.

Energy momentum tensor

Energy-momentum tensor:

$$T[\psi]_{\alpha\beta} = \nabla_\alpha\psi\nabla_\beta\psi - g_{\alpha\beta}(\nabla_\gamma\psi\nabla^\gamma\psi).$$
$$^{(X)}P[\psi]_\alpha = T[\psi]_{\alpha\beta}X^\beta,$$
$$E_X[\psi](\Sigma) = \int_\Sigma {}^{(X)}P[\psi]_\alpha \mathrm{d}\nu^\alpha.$$

Properties:

1. $\mathbf{T}$ timelike, $\implies E_{\mathbf{T}} \geq 0$.
2. S a symmetry, $\implies$ Same properties for $E_X[S\psi]$.
   (generalised symmetry S: $\nabla^\alpha\nabla_\alpha\psi = 0$ gives $\nabla^\alpha\nabla_\alpha S\psi = 0$)
3. $E_X(t_2) - E_X(t_1) = \int T[\psi]_{\alpha\beta}\nabla^{(\alpha}X^{\beta)}\mathrm{d}^4\mu_g$.

Model black holes

[Figure goes here]

- Some null geodesics terminate on i^+.
- Assume $\mathcal{M} \sim \mathbb{R} \times (r_+, \infty) \times S^2$.
- In stationary spacetime, consider **null orbits** -null geodesics which, projected in the quotient, are constant or periodic.
- In Schwarzschild, null orbits at $r = 3M$. In Kerr, at $3M + O(a)$. All unstable.

Geometry of Schwarzschild and Kerr

- Mass M, rotational parameter a.
- Schwarzschild is $a = 0$, subcritical Kerr is $|a| < M$.
- Spherical co-ordinates, (t, r, θ, ϕ):

$$\begin{aligned} g =& -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 \\ &+ \Sigma d\theta^2 + \left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right)\frac{\sin^2\theta}{\Sigma} d\phi^2, \\ \Sigma =& r^2 + a^2\cos^2\theta, \\ \Delta =& r^2 - 2Mr + a^2. \end{aligned}$$

- Exterior: $r > r_+ = M + \sqrt{M^2 - a^2}$.
- Symmetries: ∂_t, ∂_ϕ.

Kerr Problems

Problems:

1. No timelike, Killing vector: no positive, conserved energy.
2. $\partial_t, \partial_\phi$ only Killing vectors:
   $E_{\mathbf{T}}[k^n u]$ doesn't control Sobolev norms,
3. Photon orbits: "trapping".
   Can't prove Morawetz/ local energy estimate using a vector field.

Energy

Use blended energy

- Stationary vector field timelike for r large ∂_t.
- Null generator extension timelike for r near r_+ $\partial_t + \omega_H \partial_\phi$.
- For $|a|$ small overlap.

Let

$$T_\chi = \partial_t + \chi \omega_H \partial_\phi.$$

Timelike in full exterior.
Failure to be conserved controlled by Morawetz (local energy) estimate.

Hidden symmetries

Hidden symmetry from Carter Killing 2-tensor

$$\mathcal{Q} = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{\cos^2\theta}{\sin}\theta\partial_\phi^2 + a^2\sin^2\theta\partial_t^2 .$$

Symmetry algebra

$$\mathbb{S}_2 = \{S_{\underline{a}}\}_{\underline{a}} = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, \mathcal{Q}\},$$

$$|\Delta_{S^2} u|^2 \leq |\mathcal{Q}u|^2 + |\partial_t^2 u|^2 + |\partial_\phi^2 u|^2,$$

$$E_{\mathbf{T}}[\Delta_{S^2} u] \leq \sum_{\underline{a}} E_{\mathbf{T}}[S_{\underline{a}} u].$$

Higher energies and momenta for $\mathbb{S}_2$ vectors

Let

$$E_{X,n+1}[\psi] = \sum_{i=0}^{n} \sum_{S \in \mathbb{S}_i} E_X[S\psi],$$

$$|\psi|_n^2 = \sum_{i=0}^{n} \sum_{S \in \mathbb{S}_i} |S\psi|^2.$$

Let

$$T[\psi_1, \psi_2]_{\alpha\beta} = (1/4)\left(T[\psi_1 + \psi_2]_{\alpha\beta} - T[\psi_1 - \psi_2]_{\alpha\beta}\right),$$

$$T[\psi_1]_{\underline{ab}\alpha\beta} = T[S_{\underline{a}}\psi, S_{\underline{b}}\psi]_{\alpha\beta}.$$

Given $X^{\underline{ab}}$, let

$${}^{(X^{\underline{ab}})}P[\psi]_\alpha = T[\psi_1]_{\underline{ab}\alpha\beta} X^{\underline{ab}\beta},$$

$$E_{X^{\underline{ab}}}[\psi] = \int {}^{(X^{\underline{ab}})}P[\psi]_\alpha \mathrm{d}\nu^\alpha.$$

Morawetz estimate

Wave equation takes form $\left(\partial_r \Delta \partial_r - \Delta^{-1}\mathcal{R}(r)^{\underline{a}} S_{\underline{a}}\right)\psi = 0$.
$(\Delta = r^2 - 2Mr + a^2)$

Build $A^{\underline{ab}}$ from $\mathcal{R}^{\underline{a}}$, $\mathcal{L}^{\underline{b}}$, and weights z and w, where

$$\mathcal{L}^{\underline{a}} S_{\underline{a}} = \partial_t^2 + \partial_\phi^2 + \mathcal{Q}.$$

Suitable choices $\implies E_{T_\chi,3} \geq C E_{A^{\underline{ab}}}$.
(Lower-order terms)

Morawetz estimate (cont.)

Get $T_{\underline{ab}\alpha\beta}\nabla^\alpha A^{\underline{ab}\beta}$ like

$$\Delta^{3/2}z^{1/2}\left(\partial_r w\frac{z^{1/2}}{\Delta^{1/2}}\left(-\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\right)(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)$$
$$+ w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)$$
$$+ \frac{1}{4}\left(\partial_r\Delta\partial_r z\left(\partial_r w\left(\partial_r\frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\right)\right)\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).$$

So

$$E_{T_\chi,3}(t_2)+E_{T_\chi,3}(t_1)$$
$$\geq C\int\frac{1}{r^2}|\partial_r\psi|_2^2+\mathbf{1}_{r\not\approx 3M}\frac{1}{r^3}\left(|\partial_t\psi|_2^2+|\not\nabla\psi|_2^2\right)+\frac{1}{r^4}|\psi|_2\mathrm{d}^4\mu_g.$$

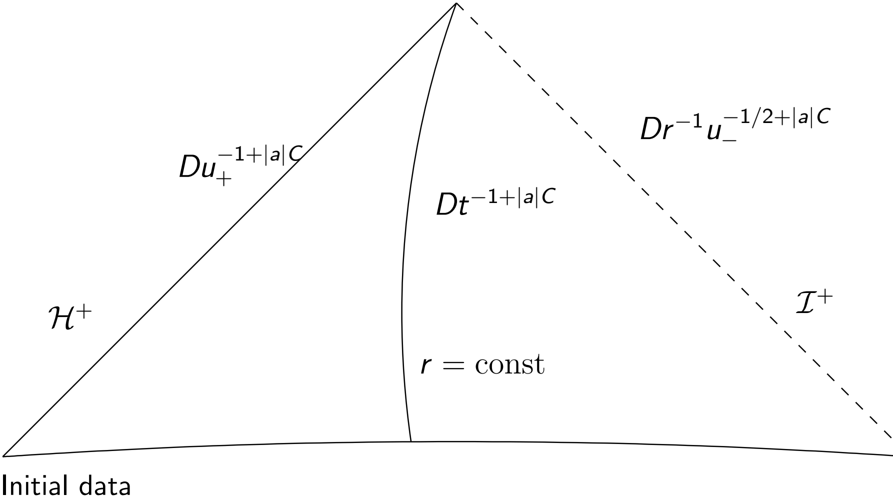
Bounded energy argument and local energy decay

$$E_{T_\chi,3}(t_2) - E_{T_\chi,3}(t_1) \geq |a| C \int ((\mathit{localisation}))|\partial^3\psi|^2 \mathrm{d}^4\mu_g$$
$$\geq |a| C (E_{T_\chi,3}(t_2) + E_{T_\chi,3}(t_1)).$$

$$E_{T_\chi,3}(t_2) \leq \frac{1+|a|C}{1-|a|C} E_{T_\chi,3}(t_1)$$

$$E_{T_\chi,3}(t_1) \geq$$
$$C \int \frac{1}{r^2}|\partial_r\psi|_2^2 + \mathbf{1}_{r \not\approx 3M} \frac{1}{r^3}\left(|\partial_t\psi|_2^2 + |\not\nabla\psi|_2^2\right) + \frac{1}{r^4}|\psi|_2 \mathrm{d}^4\mu_g$$

Theorem: On Kerr with $|a| \ll M$, $u_\pm = t \pm (r + r_+ \log(r/rp - 1))$



$$D^2 = \|\psi\|(0)^2 = E_{T_\chi,9}(0) + E_{K,5}(0) + E_{n,3}(0).$$