Decay for the wave equation outside a slowly rotating Kerr black hole

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Hidden symmetries and Kerr wave decay

Joint with Lars Andersson.

- Kerr spacetime
- Wave: $\nabla^\alpha \nabla_\alpha \psi = 0$, decoupled, important equation, model.
- Goal: robust tools (hopefully) for Kerr stability.
- We consider $|a| \ll M$, exterior $r > r_+$.
- Result: $t^{-1+|a|C}$ decay for $|a| \ll M$ [arXiv:0908.2265].

Minkowski stability

Minkowski $\mathbb{R}^{3+1}, -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2$.

- Friedrich
- Christodoulou- Klainerman
- Lindblad- Rodnianski

Kerr wave decay: other results

- $|a| \leq M$, mode decay: Finster-Kamran-Smoller-Yau
- $|a| \ll M$:
  Boundedness, **integrated decay/ Morawetz estimate**, decay rates.
  Dafermos-Rodnianski, Tataru-Tohaneanu, Tataru (t^{-3})
- (Builds on earlier Schwarzschild Morawetz and conformal energy results: Łaba- Soffer, B- Soffer, B- Sterbenz, Dafermos- Rodnianski, Metcalfe- Marzuola- Tataru- Tohaneanu, Luk. See also Donninger- Schlag- Soffer.)
- spectral and scattering results.

Morawetz/ local energy estimates

For some localising weight $\mathbf{1}_A$ independent of t,

$$\int_0^\infty \int_{\text{space}} \mathbf{1}_A |\partial\psi|^2 \mathrm{d}^3x \mathrm{d}t \leq C \int_{\text{space}} |\partial\psi|^2 \mathrm{d}^3 x.$$

Any vector field which generates a smooth symmetry of a Lagrangian generates, for each solution of the associated Euler-Lagrange equations, a quantity which is conserved.

Energy-momentum tensor

Energy-momentum tensor:

$$T[\psi]_{\alpha\beta} = \nabla_\alpha\psi\nabla_\beta\psi - g_{\alpha\beta}(\nabla_\gamma\psi\nabla^\gamma\psi).$$

Given a vector-field X, 4-momentum

$$\begin{aligned} {}^{(X)}P[\psi]_\alpha &= T[\psi]_{\alpha\beta}X^\beta, \\ E_X[\psi](\Sigma) &= \int_\Sigma {}^{(X)}P[\psi]_\alpha \mathrm{d}\nu^\alpha. \end{aligned}$$

Assume spacetime is foliated by Σ_t with timelike, future-oriented normal

$$E_X[\psi](\Sigma_t) = E_X[\psi](t) = E_X[\psi] = E_X(t)$$

Energy-momentum properties

Properties:

1. If $\mathbf{T}$ timelike,

   then $E_{\mathbf{T}} \geq 0$.

2. We call S a (generalised) symmetry when
   $\nabla^\alpha \nabla_\alpha \psi = 0 \implies \nabla^\alpha \nabla_\alpha S\psi = 0$,
   If S is a symmetry,

   then $E_X[S\psi]$ has the same properties as $E_X[\psi]$.

3. If X is Killing,

   then $E_X(t_2) = E_X(t_1)$.

   Otherwise: $E_X(t_2) - E_X(t_1) = \int T[\psi]_{\alpha\beta} \nabla^{(\alpha} X^{\beta)} \mathrm{d}^4\mu_g$.

Geometry of Schwarzschild and Kerr

- Mass M, rotational parameter a.
- Schwarzschild is $a = 0$, subcritical Kerr is $|a| < M$.
- Spherical co-ordinates, (t, r, θ, ϕ):

$$
\begin{aligned}
g =& -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dtd\phi + \frac{\Sigma}{\Delta} dr^2 \\
&+ \Sigma d\theta^2 + \left((r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right) \frac{\sin^2\theta}{\Sigma} d\phi^2, \\
\Sigma =& r^2 + a^2\cos^2\theta, \\
\Delta =& r^2 - 2Mr + a^2.
\end{aligned}
$$

- Exterior: $r > r_+ = M + \sqrt{M^2 - a^2}$.
- Symmetries: ∂_t, ∂_ϕ.

Ergoregion

$$E_{\partial_t}[\psi](\Sigma_t) = \frac{1}{2}\int_{\Sigma_t}\Big(\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Delta}(\partial_t\psi)^2 + 4aMr(\partial_t\psi)(\partial_\phi\psi) + \Delta(\partial_r\psi)^2 + (\partial_\theta\psi)^2 + \frac{\Delta - a^2}{\Delta}(\partial_\phi\psi)^2\Big) \sin\theta \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r.$$

Not everywhere time-like vector field:

$$g(\partial_t,\partial_t) = \frac{r^2 - 2Mr + a^2\cos^2\theta}{r^2 + a^2\cos^2\theta}.$$

Trapping

Null geodesics γ:
Conserved quantities: $E = g(\dot\gamma, \partial_t)$, $L_z = g(\dot\gamma, \partial_\phi)$, $0 = g(\dot\gamma, \dot\gamma)$,
and Carter constant:

$$\mathcal{Q} = g(\dot\gamma, \partial_\theta)^2 + \cot^2\theta L_z^2 + a^2 \sin^2\theta E^2.$$

Complete separation:

$$\Sigma^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \mathcal{R}(r; M, a; E, L_z, Q),$$

$$\mathcal{R}(r; M, a; E, L_z, Q) = -(r^2 + a^2)^2 E^2 - (4aMr)EL_z + \Delta\mathcal{Q} + (\Delta - a^2)L_z^2. \quad (1)$$

ODE analysis, turning points, orbits when $\mathcal{R} = \partial_r\mathcal{R} = 0$,
hyperbolic orbits $\partial_r^2\mathcal{R} < 0$, at $r = 3M + O(a)$. c.f. (2)

Kerr Problems

Problems:

1. No timelike, Killing vector: no positive, conserved energy.
2. $\partial_t, \partial_\phi$ only Killing vectors:
   $E_{\mathbf{T}}[k^n u]$ doesn't control Sobolev norms,
3. Photon orbits: "trapping".
   Can't prove Morawetz/ local energy estimate using a vector field.

Energy

Use blended energy

- Stationary vector field timelike for r large ∂_t.
- Null generator extension timelike for r near r_+ $\partial + \omega_H \partial_\phi$.
- For $|a|$ small overlap.

Let

$$T_\chi = \partial_t + \chi \omega_H \partial_\phi.$$

Timelike in full exterior.
Failure to be conserved controlled by Morawetz (local decay) estimate.

The wave equation

The wave equation:

$$\left(\partial_r \Delta \partial_r + \frac{1}{\Delta}\mathcal{R}(r; M, a; \partial_t, \partial_\phi, \mathcal{Q})\right)\psi = 0 \tag{2}$$

where $\Delta = r^2 - 2Mr + a^2$ and

$$\mathcal{Q} = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \cot^2\theta\partial_\phi^2 + a^2\sin^2\theta\partial_t^2.$$

c.f. (1).

Symmetry algebra

The set of symmetries forms a graded algebra.
Iterated Lie derivatives along Killing vectors are symmetries.
In Kerr, there are Killing vectors

$$\partial_t, \qquad \partial_\phi.$$

There is also the (hidden) symmetry:

$$\mathcal{Q}$$

Order n generators:

$$\mathbb{S}_n = \{\partial_t^{n_t} \partial_\phi^{n_\phi} \mathcal{Q}^{n_Q} \mid n_t, n_\phi, n_Q \in \mathbb{N}, n_t + n_\phi + 2n_Q = n\},$$
$$\mathbb{S}_0 = \{\mathrm{Id}\}, \ \mathbb{S}_1 = \{\partial_t, \partial_\phi\}, \ \mathbb{S}_2 = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, \mathcal{Q}\} = \{S_{\underline{a}}\}.$$

Higher energies

Let

$$E_{X,n+1}[\psi] = \sum_{i=0}^{n} \sum_{S \in \mathbb{S}_i} E_X[S\psi],$$
$$|\psi|_n^2 = \sum_{i=0}^{n} \sum_{S \in \mathbb{S}_i} |S\psi|^2.$$

Thus,

$$|\Delta_{S^2}\psi| \leq |\mathcal{Q}\psi| + |\partial_\phi^2\psi| + |\partial_t^2\psi| \leq |\psi|_2,$$

and for $r > r_0 > r_+$,

$$|\psi(t, r, \theta, \phi)| \lesssim E_{\mathbf{T},3}(t).$$

Morawetz (local energy) estimate idea

$$A = \mathcal{F}\partial_r$$

(additional terms)
Illustrate method by integration by parts:

$$\begin{aligned}
0 =& (\mathcal{F}\partial_r\psi)(\partial_r^2\psi + \mathcal{R}\psi) \\
=& (\partial_r\psi)\frac{1}{2}(\mathcal{F}')(\partial_r\psi) + \psi(-\mathcal{F})\left(\partial_r\mathcal{R}\right)\psi \\
& + \text{l.o.t.s} \\
& + \partial_t(\mathcal{F}\psi'\partial_t\psi) + \partial_r((1-2M/r)(\text{terms})).
\end{aligned}$$

$\Delta\mathcal{F}$ bounded $\implies \int_{\Sigma_t} |\Delta\mathcal{F}(\partial_r\psi)\partial_t\psi| \mathrm{d}^3\mu \leq E_{\mathbf{T}}$.

Higher energies and momenta for $\mathbb{S}_2$ vectors

Let

$$T[\psi_1,\psi_2]_{\alpha\beta} = (1/4)\left(T[\psi_1+\psi_2]_{\alpha\beta} - T[\psi_1-\psi_2]_{\alpha\beta}\right),$$
$$T[\psi_1]_{\underline{ab}\alpha\beta} = T[S_{\underline{a}}\psi, S_{\underline{b}}\psi]_{\alpha\beta}.$$

Given $X^{\underline{ab}}$, let

$$^{(X^{\underline{ab}})}P[\psi]_\alpha = T[\psi]_{\underline{ab}\alpha\beta} X^{\underline{ab}\beta},$$
$$E_{X^{\underline{ab}}}[\psi] = \int {}^{(X^{\underline{ab}})}P[\psi]_\alpha \mathrm{d}\nu^\alpha.$$

Morawetz estimate

Wave equation takes form $\left(\partial_r \Delta \partial_r - \Delta^{-1} \mathcal{R}(r)^{\underline{a}} S_{\underline{a}}\right) \psi = 0$.
$(\Delta = r^2 - 2Mr + a^2)$

Build $A^{\underline{ab}}$ from $\mathcal{R}^{\underline{a}}$, $\mathcal{L}^{\underline{b}}$, and weights z and w, where

$$\begin{aligned} A^{\underline{ab}} &= zw(\partial_r \mathcal{R}^{(\underline{a}}) \mathcal{L}^{\underline{b})}, \\ \mathcal{L}^{\underline{a}} S_{\underline{a}} &= \partial_t^2 + \partial_\phi^2 + \mathcal{Q}. \end{aligned}$$

Suitable choices $\implies$ $E_{T_\chi, 3} \geq C E_{A^{\underline{ab}}}$.
(Lower-order terms)

Morawetz estimate (cont.)

Get $T_{\underline{ab}\alpha\beta}\nabla^\alpha A^{\underline{ab}\beta}$ like

$$\Delta^{3/2} z^{1/2}\left(\partial_r w \frac{z^{1/2}}{\Delta^{1/2}}\left(-\partial_r \frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\right)(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)$$
$$+ w\left(\partial_r \frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)\left(\partial_r \frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)$$
$$+ \frac{1}{4}\left(\partial_r \Delta \partial_r z\left(\partial_r w\left(\partial_r \frac{z}{\Delta}\mathcal{R}^{\underline{b}}\right)\right)\right)\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi). \tag{3}$$

So (c.f. 24)

$$E_{T_\chi,3}(t_2)+E_{T_\chi,3}(t_1) \geq C\int \frac{1}{r^2}|\partial_r\psi|_2^2 + \mathbf{1}_{r\not\approx 3M}\frac{1}{r^3}\left(|\partial_t\psi|_2^2 + |\not\nabla\psi|_2^2\right) + \frac{1}{r^4}|\psi|_2 \mathrm{d}^4\mu_g .$$

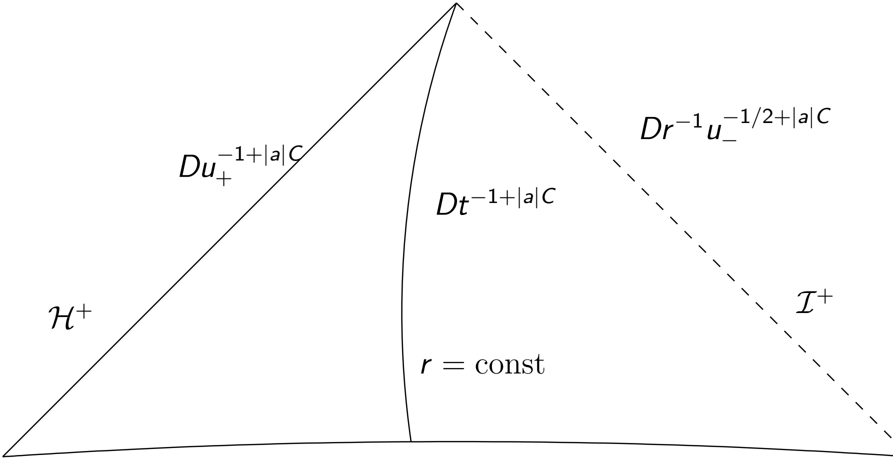
Bounded energy argument and local energy decay

$$E_{T_\chi,3}(t_2) - E_{T_\chi,3}(t_1) \geq |a| C \int ((\mathit{localisation}))|\partial^3\psi|^2 \mathrm{d}^4\mu_g$$
$$\geq |a| C (E_{T_\chi,3}(t_2) + E_{T_\chi,3}(t_1)).$$

$$E_{T_\chi,3}(t_2) \leq \frac{1+|a|C}{1-|a|C} E_{T_\chi,3}(t_1)$$

$$E_{T_\chi,3}(t_1) \geq$$
$$C \int \frac{1}{r^2}|\partial_r\psi|_2^2 + \mathbf{1}_{r \not\approx 3M} \frac{1}{r^3}\left(|\partial_t\psi|_2^2 + |\not\nabla\psi|_2^2\right) + \frac{1}{r^4}|\psi|_2 \mathrm{d}^4\mu_g$$

Theorem: On Kerr with $|a| \ll M$, $u_\pm = t \pm (r + r_+ \log(r/rp - 1))$



Initial data

$$D^2 = \|\psi\|(0)^2 = E_{T_\chi,9}(0) + E_{K,5}(0) + E_{n,3}(0).$$

Maxwell equation

Maxwell equation:

$$F_{\alpha\beta} = F_{[\alpha\beta]}, \qquad \nabla^\alpha F_{\alpha\beta} = 0, \qquad \nabla_{[\alpha} F_{\beta\gamma]} = 0.$$

Spin-lowering

$$\begin{aligned} \phi_0 &\sim *F(\Theta,\Phi) + iF(\Theta,\Phi), \\ \Theta &= \partial_\theta, \\ \Phi &= \frac{1}{\sin\theta}\partial_\phi + a\sin\theta\partial_t, \\ \frac{1}{\sin\theta}\Theta\sin\theta\Theta + \frac{1}{\sin^2\theta}\Phi\Phi &= \mathcal{Q} + \partial_\phi^2 + 2\partial_\phi\partial_t. \end{aligned}$$

Ipser-Fackerel equation:

$$(\Box + V)((r - ia\cos\theta)^{-1}\phi_0) = 0.$$

Maxwell (cont.)

There are explicitly known stationary solutions.
In Schwarzschild, these are spherically symmetric.

The extra potential V requires a stronger Hardy estimate. c.f. (3).
In Schwarzschild, project out $l = 0$ stationary solution, and use $|\nabla\!\!\!/\psi|^2 > |\psi|^2$ to get more positivity so the Hardy estimate is sufficient.