Decay for the Maxwell field on the Schwarzschild manifold

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Black hole stability

- Price: Schwarzschild; wave, Maxwell, Einstein: t^{-3-2l}.
- Wald: Schwarzschild; wave: bounded.
- Christodoulou-Klainerman: $\mathbb{R}^{1+3}$; wave, Maxwell, Einstein: $t^{-3/2}$, $t^{-5/2}$, $t^{-7/2}$.
- Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski: Schwarzschild, wave: t^{-1}.

Vast literature: Finster-Kamran-Smoller-Yau, Dimock-Kay, Whiting, Bachelot, Häfner, Nicolas.

Outline

- Conserved (generalised) energy.
- Sobolev estimate

$$|f(x_0)| < C\left(\int |f|^2 + |\nabla^2 f|^2 d^3x\right)^{1/2}.$$

- No spherical harmonic decomposition.

Schwarzschild manifold

$$ds^2 = (1 - 2M/r)(-dt^2 + (1 - 2M/r)^{-2} dr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

$$dr_* = (1 - 2M/r)^{-1} dr,$$
$$r = 3M \iff r_* = 0.$$

Symmetries $\mathbb{T} = \{T, \Theta_i\}$.

$r = 3M$ photon sphere, orbiting geodesics $\Rightarrow$ **trapping**.

Maxwell equations

$$
\begin{aligned}
F_{[\alpha\beta]} =& 0. \\
\nabla_{[\gamma} F_{\alpha\beta]} =& 0, \\
\nabla^{\alpha} F_{\alpha\beta} =& 0.
\end{aligned}
\tag{1}
$$

- Decoupled problem.
- Initial data

$$
F_{\alpha\beta}(\{0\} \times \mathbb{R} \times S^2) = (F_0)_{\alpha\beta} \tag{2}
$$

(with constraint).

Spinor decomposition

$$\begin{aligned}\hat{l} =&(1-2M/r)^{1/2}(\partial_t + \partial_{r_*}),\\ \hat{n} =&(1-2M/r)^{1/2}(\partial_t - \partial_{r_*}),\\ m =&r^{-1}(\partial_\theta + sin(\theta)^{-1}\partial_\phi).\end{aligned}$$

$$\begin{aligned}\phi_1 =&F(\hat{l}, m)\\ \phi_0 =&\frac{1}{2}(F(\hat{l}, \hat{n}) + iF(m, \bar{m}))\\ \phi_{-1} =&F(\hat{n}, \bar{m}),\end{aligned}$$

$$\begin{aligned}u_+ =&t + r_*,\\ u_- =&t - r_*.\end{aligned}$$

Vector fields: "conserved quantities"

Energy-momentum tensor:

$$\mathbf{T}_{\alpha\beta}.$$

Energy:

$$T = \partial_t,$$

$$\begin{aligned}
&E_T[F](t) \\
&= \int_{\{t\}\times\mathbb{R}\times S^2} \mathbf{T}_{\alpha\beta} T^\alpha d\nu^\beta \\
&= \int_{\{t\}\times\mathbb{R}\times S^2} \left(\frac{|\phi_1|^2}{4} + \frac{|\phi_0|^2}{2} + \frac{|\phi_{-1}|^2}{4} \right) (1 - 2M/r) r^2 dr_* d^2\omega,
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_T[F](0) = 0.$$

$$K = (t^2 + {r_*}^2)\partial_t + 2tr_*\partial_{r_*}$$

$$E_K[F](t) = \frac{1}{4}\int \left(u_+^2|\phi_1|^2 + (u_+^2 + u_-^2)|\phi_0|^2 + u_-^2|\phi_{-1}|^2\right)(1 - 2M/r)dr_* d^2\omega.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_K[F](t) = \int_{\{t\}\times\mathbb{R}\times S^2} t \underbrace{\left(1 - \frac{r_*}{r}\left(1 - \frac{3M}{r}\right)\right)}_{\text{Positive in compact set}} |\phi_0|^2(1 - 2M/r)r^2 dr_* d^2\omega.$$

Spin reduction: ϕ_0 satisfies a wave equation.

Sobolev estimates

- Sobolev estimate:

$$|f(\vec{x})| \leq \left(\int_{\mathbb{R}^3} |f|^2 + |\nabla^2 f|^2 d^3x \right)^{1/2}.$$

- Θ_i is a symmetry $\Rightarrow$ if F solves the Einstein equation, so does $\mathcal{L}_{\Theta_i} F$.
- **Trade derivatives:** use the Maxwell equation and control over T, Θ_i derivatives to control ∂_{r_*} derivatives.

Details

Spin reduction:

$$\sum_{k=0}^{n} E_K[\mathcal{L}_{\mathbb{T}}^k F](0) \leq C \left(\sum_{k=0}^{n+1} E_K[\mathcal{L}_{\mathbb{T}}^k F](0) + \sum_{k=0}^{n+5} E_T[\mathcal{L}_{\mathbb{T}}^k F](0) \right).$$

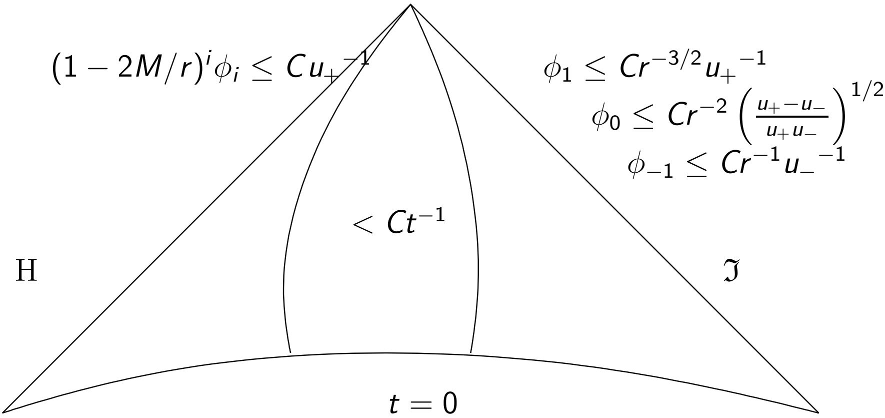
Derivative trading:

$$\mathbb{X} = \{T, R, \Theta, \Phi\},$$

$$|F|_{\mathbb{X},n,\mathbb{T}} = \sum_{k=0}^{n} \sum_{X_j \in \mathbb{X}} |\mathcal{L}_{\mathbb{T}}^k F(X_1, X_2)|,$$

$$\sum_{X_j \in \mathbb{X}} |\mathcal{L}_{\mathbb{T}}^n \mathcal{L}_R F(X_1, X_2)| \leq |F|_{\mathbb{X},n+1,\mathbb{T}}.$$

Main result: Decay estimates



Constant depends on energies of F and 8 derivatives. Initial data need not be restricted to finitely many spherical harmonics or to vanish on the event horizon.

Future work

- Reach conjectured decay
- Yang-Mills and linearized gravity on Schwarzschild
- Kerr

Discussing with Lars Andersson (AEI) and Nikodem Szpak (AEI).