Decay for the Maxwell field on the Schwarzschild manifold

Pieter Blue

AEI, 18 September 2007

Schwarzschild manifold

$$ds^2 = (1 - 2M/r)(-dt^2 + (1 - 2M/r)^{-2} dr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

$$dr_* = (1 - 2M/r)^{-1} dr,$$

$$r = 3M \Longleftrightarrow r_* = 0.$$

Maxwell equations

$$\begin{aligned} F_{[\alpha\beta]} &= 0. \\ \nabla_{[\gamma} F_{\alpha\beta]} &= 0, \\ \nabla^{\alpha} F_{\alpha\beta} &= 0. \end{aligned} \tag{1}$$

- Decoupled problem.
- Initial data

$$F_{\alpha\beta}(\{0\} \times \mathbb{R} \times S^2) = (F_0)_{\alpha\beta} \tag{2}$$

(with constraint).

Sets of vectors

$$
\begin{aligned}
T &= \frac{\partial}{\partial t}, & R &= \frac{\partial}{\partial r_*}, \quad \Theta = \frac{\partial}{\partial \theta}, \quad \Phi = \frac{\partial}{\partial \phi} \\
\hat{T} &= (1 - 2M/r)^{-1/2} \frac{\partial}{\partial t}, & \hat{R} &= \ldots
\end{aligned}
$$

Bases

$$
\begin{aligned}
\mathbb{X} &= \{T, R, \Theta, \Phi\}, \\
\hat{\mathbb{X}} &= \{\hat{T}, \hat{R}, \hat{\Theta}, \hat{\Phi}\}.
\end{aligned}
$$

Symmetries

$$
\begin{aligned}
\mathbb{O} &= \{\Theta_i\}, \\
\mathbb{T} &= \{T, \Theta_i\}.
\end{aligned}
$$

Electric and magnetic components

$$\begin{aligned}\vec{E}_X &= F_{\hat{T}X} & X \in \{\hat{R}, \hat{\Theta}, \hat{\Phi}\},\\ \vec{B}_X &= F_{YZ} & X, Y, Z \text{ a cyclic permutation of } \hat{R}, \hat{\Theta}, \hat{\Phi}.\end{aligned}$$

$$\begin{aligned}|\vec{E}|^2 &= |\vec{E}_{\hat{R}}|^2 + |\vec{E}_{\hat{\Theta}}|^2 + |\vec{E}_{\hat{\Phi}}|^2,\\ |\vec{B}|^2 &= |\vec{B}_{\hat{R}}|^2 + |\vec{B}_{\hat{\Theta}}|^2 + |\vec{B}_{\hat{\Phi}}|^2.\end{aligned}$$

Spherically symmetric solutions

$$\vec{E}_{\hat{R}} = \frac{q_E}{r^2},$$
$$\vec{B}_{\hat{R}} = \frac{q_B}{r^2}.$$

- Only spherically symmetric solutions.
- Slow r_* decay, no t dynamics $\Rightarrow$ reject.

Null tetrad

Null tetrad

$$
\begin{aligned}
\hat{l} &= \hat{T} + \hat{R}, & \hat{n} &= \hat{T} - \hat{R}, & & r^{-1}e_A, r^{-1}e_B \\
\overleftarrow{l} &= T + R, & \overleftarrow{n} &= (1 - 2M/r)^{-1}(T - R), & & r^{-1}e_A, r^{-1}e_B \\
L &= T + R, & N &= T - R, & & e_A, e_B.
\end{aligned}
$$

Complex tetrad: replace angular piece by

$$
\begin{aligned}
r^{-1}e_A, r^{-1}e_B \mapsto & \qquad m = \hat{\Theta} + i\hat{\Phi}, \bar{m} \\
e_A, e_B \mapsto & \qquad M = \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}, \bar{M}.
\end{aligned}
$$

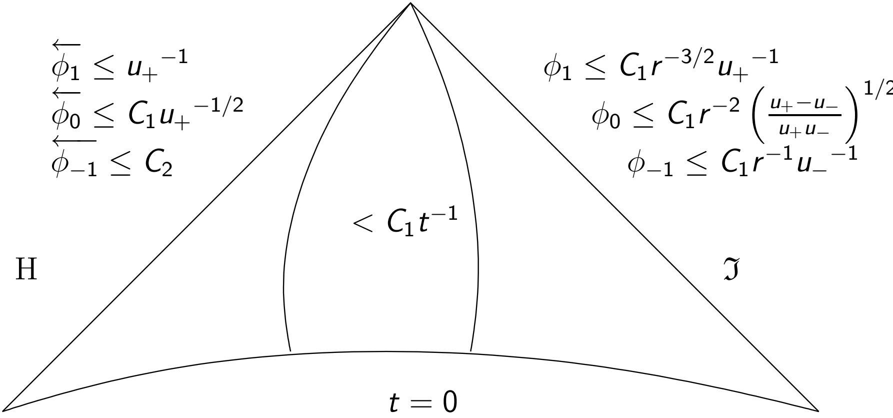
Null components

$$\begin{aligned}\phi_1 &= F(\hat{l}, m)\\ \phi_0 &= \frac{1}{2}(F(\hat{l}, \hat{n}) + iF(m, \bar{m}))\\ \phi_{-1} &= F(\hat{n}, \bar{m}),\end{aligned}$$

$$\begin{aligned}\Phi_1 &= F(L, M),\\ \Phi_0 &= \frac{1}{2}(F(L, N)(1 - 2M/r)^{-1} r^2 + F(\bar{M}, M)),\\ \Phi_{-1} &= F(N, \bar{M}).\end{aligned}$$

Decay estimates

$$u_+ = t + r_*, \qquad u_- = t - r_*.$$



Vector fields: "conserved quantities"

Matter:

$$\mathbf{T}_{\alpha\beta}.$$

For X^α,

$$
\begin{aligned}
{}^{(X)}P_\alpha &= \mathbf{T}_{\alpha\beta} X^\alpha, \\
E_X[F](\mathcal{S}) &= \int_{\mathcal{S}} {}^{(X)}P_\alpha \, d\nu^\alpha,
\end{aligned}
$$

$$E_X[F](t) = \int_{\{t\}\times\mathbb{R}\times S^2} {}^{(X)}P_\alpha \, d\nu^\alpha.$$

Vector fields: “conserved quantities” II

$$^{(X)}\pi^{\alpha\beta} = \nabla^\alpha X^\beta + \nabla^\beta X^\alpha.$$

Stokes’ Theorem

$$\int_{\partial\Omega} {}^{(X)}P_\alpha d\nu^\alpha = \frac{1}{2}\int_\Omega {}^{(X)}\pi^{\alpha\beta}\mathbf{T}_{\alpha\beta} d^4x.$$

Vector fields: “conserved quantities” III

$$\begin{aligned}\mathbf{T}(\hat{l},\hat{l}) =& |\phi_1|^2,\\ \mathbf{T}(\hat{l},\hat{n}) =& |\phi_0|^2,\\ \mathbf{T}(\hat{n},\hat{n}) =& |\phi_{-1}|^2.\end{aligned}$$

$$\begin{aligned}E_T[F](t) =& \frac{1}{2}\int_{\{t\}\times\mathbb{R}\times S^2}\sum_i |\phi_i|^2(1-2M/r)r^2 dr_* d^2\omega\\ =& E_T[F](0).\end{aligned}$$

$$\begin{aligned} K =& (t^2 + {r_*}^2)\partial_t + 2tr_*\partial_{r_*} \\ =& \frac{1}{2}({u_+}^2 L + {u_-}^2 N). \end{aligned}$$

$$4E_K[F](t) =$$

$$\int \left(u_+^2|\phi_1|^2 + (u_+^2 + u_-^2)|\phi_0|^2 + u_-^2|\phi_{-1}|^2\right)(1 - 2M/r)dr_* d^2\omega.$$

$$E_K[F](t_2) - E_K[F](t_1)$$

$$= \int_{[t_1,t_2]\times\mathbb{R}\times S^2} \underbrace{t\left(1 - \frac{r_*}{r}\left(1 - \frac{3M}{r}\right)\right)}_{\text{Positive in compact set}} |\phi_0|^2(1 - 2M/r)r^2 dr_* d^2\omega.$$

Price equations

$$
\begin{aligned}
N\Phi_1 =& M\Phi_0(1-2M/r)r^{-2},\\
L\Phi_0 =& \bar{M}\Phi_1 + \cot\theta\Phi_1,\\
N\Phi_0 =& -M\Phi_{-1} - \cot\theta\Phi_{-1},\\
L\Phi_{-1} =& -\bar{M}\Phi_0(1-2M/r)r^{-2}.
\end{aligned}
$$

Spin Reduction

- Wave equation

$$-\partial_t^2 \Phi_0 + \partial_{r_*}^2 \Phi_0 = \frac{1}{r^2}(1 - 2M/r)(-\Delta_{S^2})\Phi_0.$$

- Φ_0 controls growth of all E_K.

Wave estimates

$$E'_T[\Phi_0](t) = E'_T[\Phi_0](0),$$

$$\begin{aligned}
&E'_K[\Phi_0](t) - E'_K[\Phi_0](0) \\
&= \int_{[t,0]\times\mathbb{R}\times S^2} t'(1 - \frac{r_*}{r}(1-3M/r))|\not\nabla\Phi_0|(1-2M/r)r^2 dr_* d^2\omega dt, \\
&\leq c \sup_{t'\in[0,t]} E'_K[\Phi_0](t') + CE'_T[\Delta^2_{S^2}\Phi_0]. \qquad (3)
\end{aligned}$$

For (3), use $t\chi_{\mathrm{LC}} g\partial_{r_*}$, with χ_{LC} supported on $|r_*| < t$, $g' > 0$, and $g = 0$ at the maximum of $r^{-2}(1-2M/r)$.

$$E'_T[\Phi_0](t) = \sum_{\Theta_i} E_T[\mathcal{L}_{\Theta_i} F](0),$$
$$E'_K[\Phi_0](t) = \sum_{\Theta_i} E_K[\mathcal{L}_{\Theta_i} F](0).$$

$$E_K[F](t) \leq \sum_{\Theta_i} E_K[\mathcal{L}_{\Theta_i} F](0) + \sum_{k=0}^{5} \sum_{\Theta_i} E_K[\mathcal{L}^k_{\Theta_i} F](0).$$