Matrix factorisations and Homological Mirror Symmetry

A-model		B-model
depends on Kähler moduli	$\xleftrightarrow{\text{HMS}}$	depends on cpx str. moduli
A-branes		B-branes
Fuk		$D^b(\mathrm{Coh}\, X)$ or $\underline{\underline{MF(W)}}$

$$\mathcal{E} \xleftrightarrow{\text{Vanishing cycles}} D^b(\mathrm{Coh\ Fano}\ X)$$

$$?\ \text{whoops.} \longleftrightarrow \mathrm{Fuk}(\mathrm{Fano}\ X)$$

Motivate category of matrix factorisations

X affine e.g. $\mathbb{C}^n$

$C(X)$

Objects: bounded $\mathbb{Z}$-graded complexes of projective objects in $\mathrm{Coh}(X)$.

Morphisms: morphisms of $\mathcal{O}_X$ modules.

DG

$$H^0(\mathrm{Tw}\,\mathcal{A}) \rightsquigarrow D^b(\mathrm{Coh}\ X)$$
$$\quad\ \parallel$$
$$\quad\ CX$$

LG model

$DG_{w_0}(W) = C(X, W, w_0)$ $\qquad$ X smooth affine space

W holomorphic $X \to \mathbb{C}$

$w_0 \in \mathbb{C}$

$w_0 = 0$

Objects:

Pairs of f.g. projective $\mathcal{O}_X$ modules

$E = E_1, E_0$ with a differential

morphisms: ~~$\mathbb{Z}_2$ - differential graded category~~

$$\mathrm{Hom}(\bar{P}, \bar{Q}) = \bigoplus_{i,j} \mathrm{Hom}(P_i, Q_j)$$

$\mathrm{Pair}_{w_0}(W)$

$DB_{w_0}(W)$ ~ triangulated category

<u>Differential:</u>

$Df = q \circ f - (-1)^{k} f \circ p$

$Pair_{w_0}(W)$

Note $p^2 = W$ $q^2 = W,$ $D^2 = 0.$

$$\begin{tikzcd} P_1 \arrow[r, shift left] \arrow[d, "f_1"'] \arrow[dr, "t"] & P_0 \arrow[l, shift left] \arrow[d, "f_0"] \arrow[dl, "s"'] \\ Q_1 \arrow[r, shift left] & Q_0 \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} \overline{P} \arrow[d, "\overline{f}"] \\ \overline{Q} \end{tikzcd}$$

Morphisms in Pair

$\mathrm{Hom}(\overline{P}, \overline{Q})$

homogeneous deg 0 morphisms in $DG_{w_0}(W)$ which commute with D

Null homotopic

$f_1 = q_0 t + s p_1$

$f_0 = t p_0 + q_1 s$

Morphisms in $DB_{w_0}(W)$ = Morphisms in $Pair_{w_0}(W)$ modulo null homotopic

<u>Triangulated structure</u>

- Translation functor [1].
- Distinguished triangles satisfy axioms.

Translation functor

$T : \overline{P} \to \overline{P}[1]$

$\cancel{\overline{P}[1] = (P_0 \quad P)}$ $\quad \overline{P} = \left(P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0\right)$

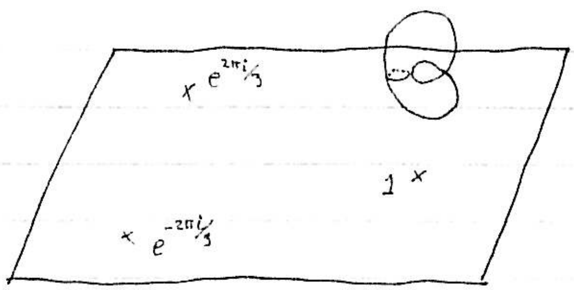
$\overline{P}[1] = \left(P_0 \underset{-p_1}{\overset{-p_0}{\rightleftarrows}} P_1\right)$

E.g. (HMS for $\mathbb{CP}^2$)

$\mathbb{CP}^2 \xrightarrow{\text{HMS}}$ LG model

$(\mathbb{C}^*)^2 \ni (x, y)$

$W = x + y + \frac{1}{xy}$



Focus on $x = y = 1$, $w_0 = 3$.

$x = u+1$

$y = v+1$

$\Rightarrow xy(x+y-3)+1=0$

$(uv)(u+v-1) + (u+v)^2 = 0$

Quadratic in u, v + higher order terms.

$u^2 + v^2$ + higher order

Holomorphic Morse Lemma $\Rightarrow$ can kill cubic terms.

LG $W = u^2 + v^2$ on $\mathbb{C}^2$.

$DB_{w_0=0}(W) =: K(n)$

$W = x_1^2 + \dots + x_n^2$

Thm: $\exists$ an equivalence of categories between

$K(n)$ and $Cl_{mod}(n)$

$\mathbb{C}$ Clifford algebra

Point: easier to compute $Cl_{mod}(n)$.

Mirror of the structure sheaf of a point (crit. value w_0)

$\rightsquigarrow$ SYZ $\rightsquigarrow$ torus in $\mathbb{CP}^2$

Clifford torus

3 local systems specified by their holonomies. $e^{2\pi i k/3}$.

$HF_\lambda(L,L)$ non-vanishing for $\lambda = e^{2\pi i k/3}$

Bonus: Action of $Cl(2,\mathbb{C})$ on HF.

$$HF_\lambda(L,L) \cong Cl(2,\mathbb{C})$$

$\mathbb{Z}_2$ coeffs

$\mathbb{Z}_2$ graded v.s.

Comment: Deformation

$$\cancel{S} \ \Omega^*(L) \cong \Lambda^*(V^\vee)$$

Grassmann algebra. $\quad$ 2 dim

<u>Graded matrix factorisations</u>

If W quasi-homogeneous

$$W(e^{i\lambda q_i} x_i) = e^{2i\lambda} W(x_i) \quad \forall \lambda \in \mathbb{R}$$

$\leftrightarrow$ vanishing of Maslov class

$\rightsquigarrow$ Floer cohomology can be $\mathbb{Z}$-graded.

$A = \bigoplus_i A_i$ graded

$$D^{gr}_{Sg}(A) := D^b(gr-A) / D^b(gr\,proj - A) \qquad \underline{\text{Orlov}}$$

f.g.

$$D^{gr}_{Sg}(A) \longrightarrow D(q\,gr - A)$$

$\rightsquigarrow$ semi-orthogonal decomposition

<u>Thm</u> (Serre): $D^b(q\,gr\,A) = D^b(\mathrm{coh}\,\mathrm{Proj}(A))$.

<u>Classical Koszul Duality</u>

$$\Lambda = \Lambda(E) = \bigoplus_{i=0}^{n+1} \Lambda^i E$$

$\mathcal{M}(\Lambda)$ - category of $\mathbb{Z}$-graded modules over Λ.

$\cup$

$\mathcal{F}$ - free graded Λ-modules

Thm $\mathcal{M}^b(\Lambda)/\mathcal{F}$ is triangulated.

Thm $D^b(\text{coh Proj } A)$ $\quad A = \mathbb{C}[x_1, \dots, x_n]$

dim $E = n$ $\quad D^b(\text{coh } \mathbb{P}^n)$.

Back to "deformation" comment:

$K(n) \cong Cl_{mod}(n)$

$D^b(\text{Coh } X) \underset{W=0}{\cong} \mathcal{M}^b(\Lambda)/\mathcal{F}$