HMS for Fano varieties

I. HMS for toric Fanos

II. $D^b(\mathbb{P}^n)$

III. $LG(\mathbb{P}^n)$.

HMS

$$X \underset{\text{mirror}}{\sim\!\sim\!\sim} Y$$

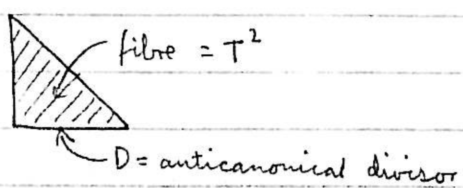
$$D\mathrm{Fuk}\, X \cong D^b \mathrm{Coh}(Y)$$

HMS for toric Fano.

X = toric variety, dim n. mirror = $(\mathbb{C}^*)^n$ $\quad W: (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ superpotential.

$X = \mathbb{P}^2 = (1 : z_1 : z_2)$

$\downarrow$



$$M = (L, \nabla)$$

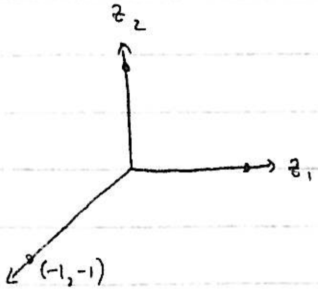
$|z_1|, |z_2|$ ← preimage of moment map = torus $\qquad S^1 \times S^1$

$$M = (\mathbb{C}^*)^2 = \{(\xi_1, \xi_2)\}$$

$$\overset{\shortparallel}{\{|z_1| \cdot e^{\mathrm{hol}_1(\nabla)}, |z_2| e^{\mathrm{hol}_2(\nabla)}\}}$$

$$W = m_2(L, \nabla)$$

$$= \sum_\beta n(\beta)\, e^{\int \beta} \mathrm{hol}_{\partial\beta}(\nabla)$$



Fan of $\mathbb{P}^2$

$$W = z_1 + z_2 + \frac{1}{z_1 z_2}$$

If Σ = fan $\quad \Sigma(1)$ = vectors spanning the fan
$= \{v_1, \dots, v_n\}$

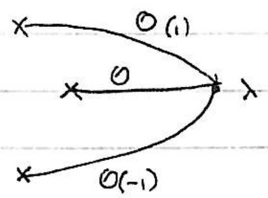
$\Rightarrow W = \sum z_i^{v_i}$ (Cho)

HMS

$D^b(X) \cong \mathcal{F}^{\rightarrow}((\mathbb{C}^*)^n, W)$ (as described in last talk)

$LG(\mathbb{P}^2) \quad (\mathbb{C}^*)^2 \xrightarrow{W} \mathbb{C}$

$$W = z_1 + z_2 + \frac{1}{z_1 z_2}$$



(this is the correspondence between vanishing cycles and $D^b(\mathrm{Coh}(\mathbb{P}^2))$.

II. $D^b \operatorname{Coh}(\mathbb{P}^n)$

$\mathcal{A}$ abelian cat.

$C^b(\mathcal{A})$ chain cpx.

$K^b(\mathcal{A})$ homotopy.

localise w.r.t. quasi-iso.

$D^b \operatorname{Coh}(\mathbb{P}^n)$ has a full strong exceptional collection

$D^b(\mathbb{P}^1)$ $\quad \mathcal{O}, \mathcal{O}(1)$

$D^b(\mathbb{P}^n)$ $\quad \mathcal{O}(k), \dots, \mathcal{O}(k+n)$

$\operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(j)[k]) = 0 \quad i < j$

<u>strong</u>: every nonzero Hom is in degree 0.

Full: generates whole category.

$\mathbb{P}^2$: $\mathcal{O}(-1), \quad \mathcal{O}, \quad \mathcal{O}(1).$

$\mathcal{O}(-1), \ L_{\mathcal{O}}\mathcal{O}(1), \ \mathcal{O}$

what is this?

$$\to L_{\mathcal{O}}\mathcal{O}(1) \to \underbrace{\operatorname{Hom}^\bullet(\mathcal{O}, \mathcal{O}(1)) \otimes \mathcal{O}}_{\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}} \to \mathcal{O}(1) \to \dots$$

$$0 \to \Omega^1 \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to 0 \to 0$$

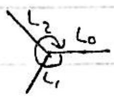
$$0 \to \Omega^1 \otimes \mathcal{O}(1) \to \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to \mathcal{O}$$

$$\to \Omega^1 \otimes \mathcal{O}(1) \to \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to$$

$$\begin{array}{ccccc} H_0 & & H_1 & & H_2 \\ \mathcal{O}(-1) & \overset{x_0}{\underset{z_0}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{y_0}}}}} & \Omega^1(1) & \overset{x_1}{\underset{z_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{y_1}}}}} & \mathcal{O} \end{array}$$

Result: $\operatorname{Hom}(H_i, H_j) \cong \Lambda^{j-i}$

$\mathcal{F}^{\to}((\mathbb{C}^*)^2, W, \succ)$

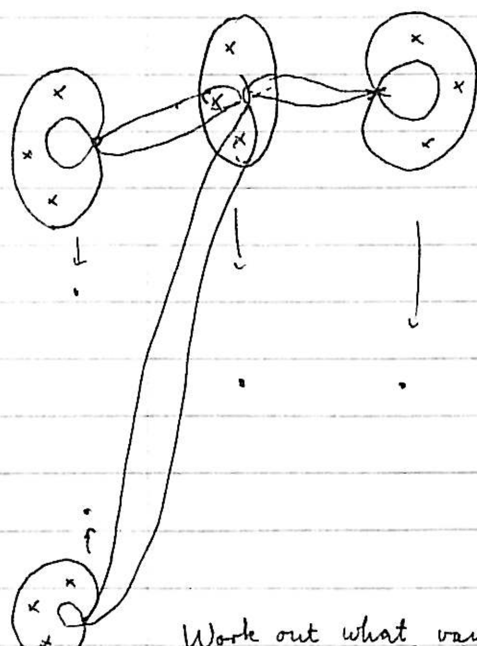


$\Sigma_\lambda = W^{-1}(\lambda)$

$\{$

$z_1 + z_2 + \frac{1}{z_1 z_2} = \lambda$

Σ_λ not singular.



Work out what vanishing cycles are. Project to $\mathbb{C}$

$\pi_\lambda : \Sigma_\lambda \longrightarrow \mathbb{C} \qquad (z_1, z_2) \longrightarrow z_1$

$\lambda = 0 \quad \pi_0 : \Sigma_0 \to \mathbb{C} \qquad (z_1, z_2) \longrightarrow z_1$

Σ_0

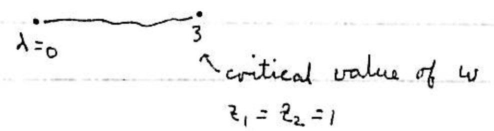
$\big\downarrow \pi$ branched cover

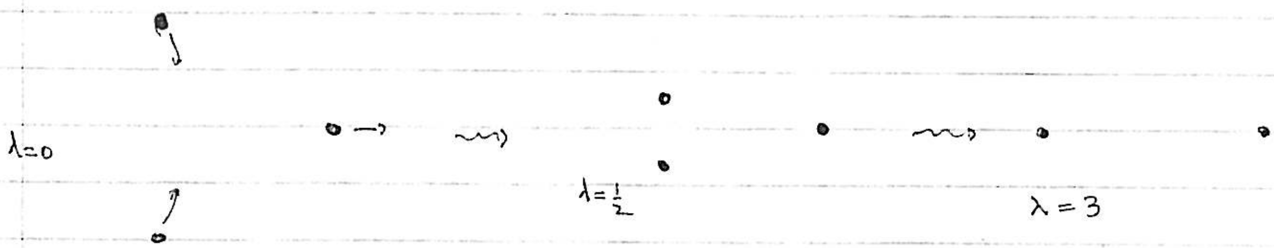
$\mathbb{C}$

× 3 × branch points

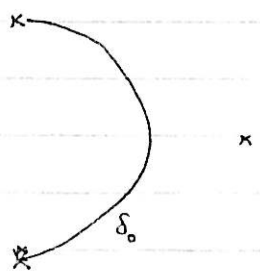
×

$\pi_\lambda : \Sigma_\lambda \longrightarrow \mathbb{C}$





$\Rightarrow$ In $\lambda = 0$, look at how branch points meet.



$\pi^{-1}(\delta_0) = L_0'$ (it's a 2-cover)

δ_0 = path traced out by 2 branch points

Claim: $L_0' \sim L_0$

Since dim $= 2$

L_0' L_0 invariant under cpx conj.

ω $(\mathbb{C}^*)^2$

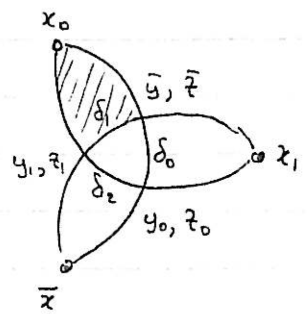
$\omega = \frac{1}{z_i} dz_i \wedge \frac{1}{\bar{z}_i} d\bar{z}_i$

$\bar{\omega} = -\omega$

L_0' Ham. isotopy $\sim L_0$ (area bounded $=$ $-$ itself $\Rightarrow$ Ham isotopic)

Σ_0

$\big\downarrow \pi_0$

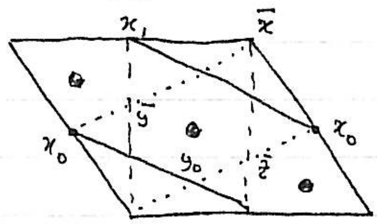


x, y, z - intersection points

Recall $x. = 1$ pt (branched)

$y, z = 2$ pts (2 cover)

$m_2(x_0, y) = \bar{z}$



$\mathrm{Hom}(L_i', L_j') = \Lambda^{j-i}$