Lefschetz fibrations

Defn: Let (M, ω) be symplectic mfd. A Lefschetz fibration is a map $f: M \longrightarrow D \; (= D^2)$ s.t.

- Crit f isolated with distinct values
- fibres of f are symplectic mfds.
- around each critical point $p \in M$ $\exists$ charts s.t.

$$f(z_1, \dots, z_n) = \sum z_i^2 \qquad (\text{in } \mathbb{C}^n \to \mathbb{C})$$

E.g. $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$

$f(z_1, z_2) = z_1^2 + z_2^2$

The fibre over $\lambda \neq 0$ looks like a cylinder 

over $\lambda = 0$ looks like pinched cylinder 

Symplectic parallel transport

Suppose $\gamma: [0,1] \longrightarrow D^2$ is a path avoiding critical values.

Want symplectomorphism

$\rho_\gamma : M_{\gamma(0)} \longrightarrow M_{\gamma(1)}$

$p \in f(\gamma(0))$

$$T_p M = T_p M_{\gamma(0)} \oplus \left(T_p M_{\gamma(0)}\right)^{\omega}$$

symplectic orthogonal complement

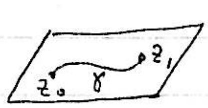
We get a canonical lift of $\frac{\partial}{\partial t}$ along γ. Now ρ_γ is the flow along this vector field.

Let $\gamma: [0,1] \longrightarrow D$ be a path from a basepoint z_0 to a critical value. γ is called a <u>vanishing path</u>.

Consider the subset

$$\Delta_\gamma := \left\{ y \in M_{\gamma(s)} : 0 \le s < 1, \lim_{t \to 1} \rho_\gamma\big|_{[s,t]}(y) = x \right\} \cup \{x\}$$

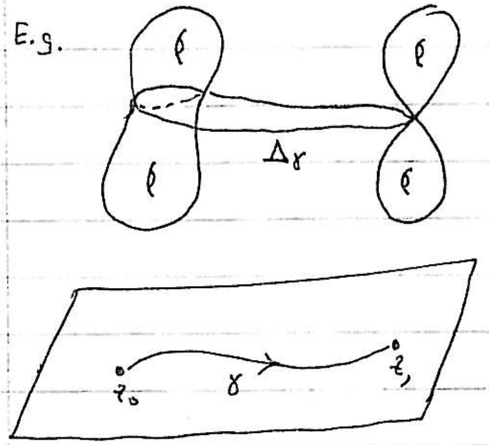
x is the unique critical point in $M_{\gamma(1)}$.



Δ_γ is called the Lefschetz thimble.

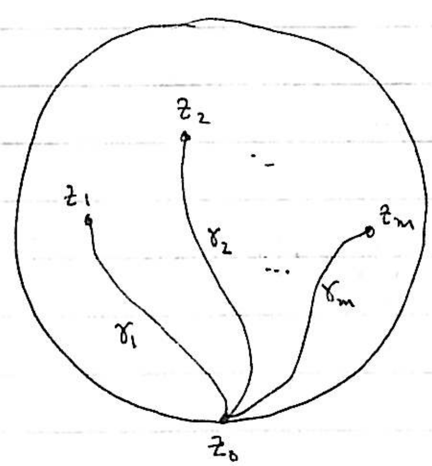
In the local model $(z_i) \mapsto \Sigma z_i^2$ this corresponds to setting the $z_i \in \mathbb{R}$.

E.g.



Let $V_\gamma = \partial \Delta_\gamma \subset M_{\gamma(0)}$. It is a Lagrangian sphere (because parallel transport gives a symplectomorphism and a point has ω vanishing).

Let $f : M \to D$ be a Lefschetz fibration. Let M_0 be the fibre at a fixed base point z_0. Think of z_0 as lying on ∂D. Choose a path ~~from~~ γ_i from z_0 to each critical value z_i. We have a collection of paths $\gamma_1, \dots, \gamma_m$ from z_0 to $z_1, \dots, z_m$.



Such a collection is called admissible if these paths are ordered by the tangent directions at z_0 clockwise.

For an admissible collection of paths the corresponding vanishing cycles $V_1, \dots, V_m$ are called a distinguished basis of vanishing cycles.

In this situation we can define the directed Fukaya category $\mathrm{Lag}^{\rightarrow}_{\{V_i\}}$ ~~...~~ (M, f)

Using the collection $\{V_1, \dots, V_m\}$

$$\mathrm{Hom}(V_i, V_j) = \begin{cases} 0 & i > j \\ \mathbb{K}\, e_j & i = j \\ CF^*(V_i, V_j) & i < j \quad (\text{in } M_{z_0}, \underline{\text{not}}\ M) \end{cases}$$

This depends on our choices of paths.

Let $\mathcal{F}(M, f) = \mathrm{Tw}\, \mathrm{Lag}^{\rightarrow}_{\{V_i\}}(M, f)$

the triangulated envelope of $\mathrm{Lag}^{\rightarrow}_{\{V_i\}}(M, f)$.

<u>Thm:</u> $\mathcal{F}(M, f)$ "The Fukaya category of the Lefschetz fibration" is an invariant of the fibration.

Note: $\{V_i\}$ form an exceptional collection for $\mathcal{F}(M, f)$.

We want to consider isotopy classes of paths from z_0 to each z_i (isotopy of path γ_i changes L_i by Ham. isotopy)

Consider $\mathcal{D} \subset \mathrm{Diff}^+(D)$ which fix z_0 and map $\{z_1, \dots, z_m\} \to \{z_1, \dots, z_m\}$

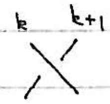
This group acts transitively on isotopy classes of paths.

Need to consider $\pi_0 \mathcal{D}$

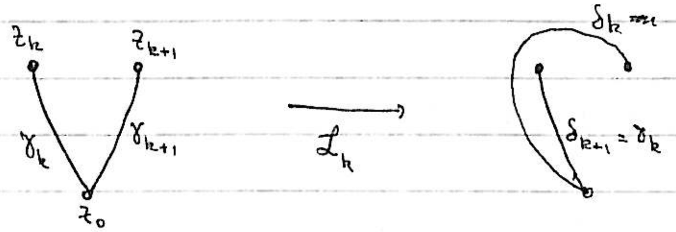
Thm: $\pi_0 \mathcal{D} = Br_m$

$\Rightarrow Br_m$ acts simply and transitively on isotopy classes of paths.

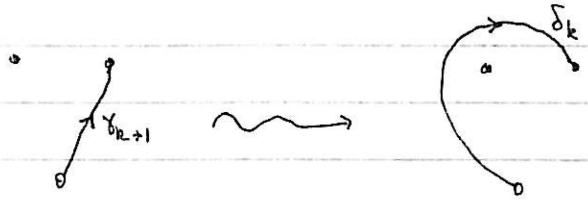
Br_m is generated by

β_k : 

How does β_k act on the paths?

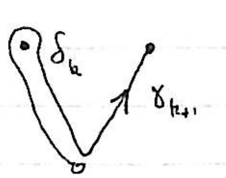


To see what happens to a distinguished basis under β_k we need to consider



Compare $V_{\gamma_{k+1}}$ to V_{δ_k}.

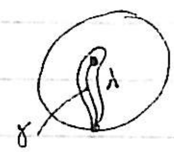
Difference between $V_{\gamma_{k+1}}$, V_{δ_k} will be the monodromy around z_k



Thm: (Symplectic hard Lefschetz theorem):

Let $f: M \to D$ be a Lefschetz fibration, γ a vanishing path, λ a loop in $D - \{\text{crit } f\}$, γ ~~winding around~~ which doubles γ winding around $\gamma(1)$. Then the monodromy around λ is Hamiltonian isotopic to a Dehn twist along V_γ

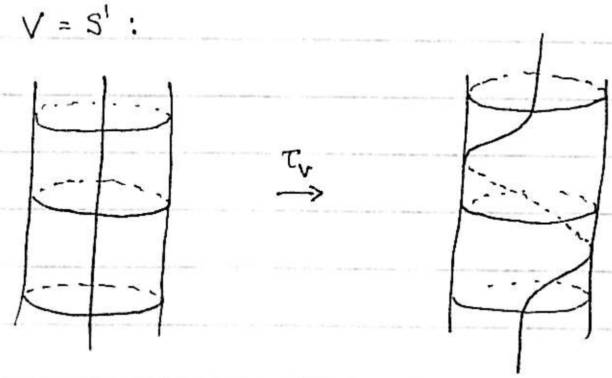
$$\text{monodromy}_\lambda = \overset{\leadsto \text{ Dehn twist}}{T_{V_\gamma}}$$



Dehn twist: M symp Mfd, $V \subset M$ Lag. sphere. Dehn twist τ_V will be supported on nbhd of V $\qquad \tau_V \in \underset{\text{symplectic}}{\mathrm{Aut}}(M, \partial M)$

$\Rightarrow$ suffices to define τ_V when $M = T^*V$.

When $V = S^1$:



Thm: Let V be a Lagrangian sphere in M, L a Lagrangian in M. Then

$$\tau_V(L) = T_V(L) \qquad \text{(Parker called this } T_V, L_V\text{)}.$$

in the Fukaya category

Cor: $\{V_1, \dots, V_m\} \longrightarrow \{V_1, \dots, V_{\delta_k}, V_{\gamma_k}, \dots, V_m\}$ are related by mutation.