<u>Derived Categories</u> (Ref: Paul's book, notes on ~~web~~ wiki)

<u>Defn</u>: A (strictly unital) A_∞ category $\mathcal{A}$ is a set of objects $Ob\,A$ and a collection of graded vec spaces $hom_{\mathcal{A}}(X_0, X_1)$ for any pair $X_0, X_1 \in Ob(A)$, with composition maps $\forall\, d \geq 1$,

$$\mu^d_{\mathcal{A}} : hom(X_{d-1}, X_d) \otimes \cdots \otimes hom(X_0, X_1) \to hom(X_0, X_d)[2-d]$$

Satisfy usual $\mathcal{A}_\infty$ equations. Also, $\exists\, e_X \in hom(X, X)$ ~~fund mor.~~ ~~...~~ s.t.

1) $\mu^1_{\mathcal{A}}(e_X) = 0$

2) $\pm \mu^2_{\mathcal{A}}(e_{X_1}, a) = a = \mu^2_{\mathcal{A}}(a, e_{X_0}) \quad \forall\, a \in hom_{\mathcal{A}}(X_0, X_1)$

3) $\mu^d(a_{d-1}, \ldots, e_{X_n}, \ldots, a_1) = 0 \quad \forall\, d > 2,\ \forall n \quad a_k \in hom(X_{k-1}, X_k)$

<u>Defn</u>: $\mathcal{A}$ = A_∞ category, $H(\mathcal{A})$ - cohomological category associated to $\mathcal{A}$.

$Ob(H(\mathcal{A})) = Ob(\mathcal{A})$

$H(hom(X_0, X_1), \mu^1_{\mathcal{A}}) = Hom_{H(\mathcal{A})}(X_0, X_1)$

Assoc. for $d=1 \Rightarrow (\mu^1)^2 = 0$

Discard all homs of deg $\neq 0$. $H^0(\mathcal{A})$.

Want to define $D(\mathcal{A})$, $\mathcal{A}$ an A_∞ category

Old

$\mathcal{A}$ abelian cat.	$\mathcal{A}$ A_∞ cat
$C^-(\mathcal{A})$, $K^-(\mathcal{A})$	$Tw\,\mathcal{A}$ twisted complexes
chain cx, htpy class of morphisms	$H^0(Tw\,\mathcal{A}) = D(\mathcal{A})$

$D(\mathcal{A})$

triangulated

<u>Defn</u>: An A_∞ functor between A_∞ categories $\mathcal{F} : \mathcal{A} \to \mathcal{B}$

- map on sets $\mathcal{F} : Ob\,\mathcal{A} \to Ob\,\mathcal{B}$
- $\forall\, d \geq 1$ $\mathcal{F}^d : hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1) \to hom_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_d)[1-d]$
- $\mathcal{F}^d$ satisfy ... relations
- $\mathcal{F}^1(e_X) = e_{\mathcal{F}(X)}$
- $\mathcal{F}^d(\ldots, e_{X_n}, \ldots) = 0 \quad \forall\, d \geq 2,\ \forall n.$

$\mu^1 \mathcal{F}^1 = \mathcal{F}^1 \mu^1$

For $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$

$H^0(\mathcal{G}) : H^0(\mathcal{A}) \longrightarrow H^0(\mathcal{B})$

For $[a]$, $H^0(\mathcal{F})[a] = [\mathcal{F}^1(a)]$

Say $\mathcal{F}$ full / faithful / quasi-equivalence if $H(\mathcal{F})$ is full/faithful/ an equivalence.

<u>Thm:</u> $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence. Then $\exists\, \mathcal{G} : \mathcal{B} \to \mathcal{A}$ a quasi-eq s.t. $\mathcal{F} \circ \mathcal{G} \simeq \mathrm{Id}_{\mathcal{B}}$

$\mathcal{G} \circ \mathcal{F} \simeq \mathrm{Id}_{\mathcal{A}}$

$$\mathcal{T}_2 : \begin{tikzcd} X_0 \arrow[loop left, "e_{X_0}"] \arrow[r, bend left, "\phi_0"] & X_1 \arrow[loop right, "e_{X_1}"] \arrow[l, bend left, "\phi_1"] \end{tikzcd}$$

$\mu^2(\phi_1, e_{X_1}) = \phi_1$

$\mu^2(e_{X_0}, \phi_1) = \phi_1$

etc.

$\mu^2(\phi_1, \phi_0) = e_{X_0}$

$\mu^2(\phi_0, \phi_1) = e_{X_1}$

Ex: $\mathcal{T}_2$ is $\mathcal{A}_\infty$ category.

<u>Defn:</u> $\mathbb{X}_0, \mathbb{X}_1 \in \mathcal{A}$ are isomorphic if $\exists\ A_\infty$ functor $\mathcal{F} : \mathcal{T}_2 \to \mathcal{A}$

$X_0 \longmapsto \mathbb{X}_0$

$X_1 \longmapsto \mathbb{X}_1$

In $H(\mathcal{A})$, $X_0 \simeq X_1$.

T_3 :
$$\begin{tikzcd} X_0 \arrow[loop above, "e_{x_0}"] \arrow[rr, "\phi_0", "\deg 0"'] & & X_1 \arrow[loop right, "e_{x_1}"] \arrow[dl, "\phi_1", "\deg 0"'] \\ & X_2 \arrow[loop right, "e_{x_2}"] \arrow[ul, "\phi_2", "\deg 1"'] & \end{tikzcd}$$

$\mu^3(\phi_2, \phi_1, \phi_0) = e_0$ (others 0)

$\mu^3(\phi_0, \phi_2, \phi_1) = e_1$

$\mu^3(\phi_1, \phi_0, \phi_2) = e_2$

Defn: An exact triangle in $\mathcal{A}$ is the image of $\mathcal{F} : T_3 \to \mathcal{A}$.

Rmk: If $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ is an A_∞ functor, $\mathcal{F} : T_3 \to \mathcal{A}$ is an exact Δ

Then $\mathcal{G} \circ \mathcal{F} : T_3 \to \mathcal{B}$ is an exact Δ in $\mathcal{B}$.

Thus any A_∞ functor is exact.

If
$$\begin{tikzcd} X_0 \arrow[rr] & & X_1 \arrow[dl] \\ & X_2 \arrow[ul, "{[1]}"] & \end{tikzcd}$$
is an exact Δ in $\mathcal{A}$, $X \in \mathcal{A}$

Then $\exists$ exact sequence

$$\mathrm{Hom}_{H(\mathcal{A})}(X, X_0) \to \mathrm{Hom}_{H(\mathcal{A})}(X, X_1) \longrightarrow \mathrm{Hom}_{H(\mathcal{A})}(X, X_2) \longrightarrow \ldots$$

exact.

$$\ldots \to \mathrm{Hom}^d(X, X_0) \to \mathrm{Hom}^d(X, X_1) \to \mathrm{Hom}^d(X, X_2) \longrightarrow \mathrm{Hom}^{d+1}(X, X_0) \to \ldots$$

and exact

$$\ldots \to \mathrm{Hom}(X_2, X) \longrightarrow \mathrm{Hom}(X_1, X) \to \mathrm{Hom}(X_0, X) \longrightarrow \ldots$$

Defn: An $\mathcal{A}_\infty$ cat. $\mathcal{A}$ is triangulated if it's "closed under shifts" and every morphism can be completed to an exact triangle.

Tw $\mathcal{A}$.

* Defn: A twisted complex in $\mathcal{A}$ is a pair (X, δ_X)

* $X = \bigoplus_{i \in I} V^i \otimes X^i$ $\quad$ V^i graded vec. spaces $\quad$ fin dim

$X^i \in \mathcal{A}$

$|I| < \infty$

* $\delta_X \in \hom^1_{Tw\mathcal{A}}(X, X)$ $\quad$ Just ~~formal tensor of homs in V^i and~~ X^i.

$$\underbrace{\sum_{d \geq 1} \mu^d(\delta_X \dots \delta_X) = 0}_{\text{gen. Maurer-Cartan eqn.}}$$

$$\hom_{Tw A}(\underset{V \otimes X}{X}, \underset{W \otimes Y}{Y}) = \bigoplus_{ij} \mathrm{Hom}(V^i, W^j) \otimes \hom_{\mathcal{A}}(X^i, Y^j)$$

$$\mu^d_{Tw\mathcal{A}}(a_d, \dots, a_1) = \sum_{i_0 \dots i_d} \mu^{d + i_0 + \dots + i_d}(\underbrace{\delta_{X_d} \dots \delta_{X_d}}_{i_d}, a_d, \delta_{X_{d-1}}, \dots, \delta_{X_{d-1}}, a_{d-1}, \dots \delta_{X_0} \dots \delta_{X_0})$$

"twisting μ^d by δ"

Fact: Tw $\mathcal{A}$ is a triangulated A_∞ category.

Defn: $D(\mathcal{A}) = H^0(Tw\,\mathcal{A})$

- triangulated category in the "old sense".

Shifts: $X = \bigoplus_{i \in I} V^i \otimes X^i$

$$SX = X[1] = \bigoplus_{i \in I} V^i[1] \otimes X^i$$

$$S\delta_X = \delta_X[1].$$

Triangles ⇝ define cones of morphisms

$f: X \to Y$ $\quad$ $\mu^1(f) = 0$ $\quad$ (we only care about completing triangles if $\mu^1(f) = 0$ so the cohomological ~~cat~~ category is triangulated)

$$\mathrm{Cone}(f) = \left(X[1] \oplus Y, \begin{pmatrix} \delta_X[1] & 0 \\ -f & \delta_Y \end{pmatrix}\right)$$

~~Cont~~

Fact:
$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl] \\ & \mathrm{Cone}(f) \arrow[ul, "{[1]}"] & \end{tikzcd}$$
is an exact triangle

$\mathcal{A}$ $\quad X \cdot \circlearrowright e_x \qquad \mu^2(e_x, e_x) = e_x$

$\mathrm{Tw}\,\mathcal{A} \ni X = V \otimes X$

graded vec. space

δ_x - differential on V

$\mathrm{Tw}\,\mathcal{A}$ = dg vec. spaces over $k \qquad e : V \to W$

$\mu^1_{\mathrm{Tw}\mathcal{A}}(f) = \delta_x f \pm f \delta_x$

~~Ho~~ Homomorphisms in $H(\mathrm{Tw}\,\mathcal{A})$ will be ~~eq~~ homotopy equivalence classes of chain maps

$\Rightarrow H^0(\mathrm{Tw}\,\mathcal{A}) = K(\mathrm{Vect}_k) = D(\mathrm{Vect}_k)$

homotopy category

* Way to think about this: a twisted complex is a collection of objects with differential δ a sequence of maps between them "going to the right" somehow. ~~The sum~~ For every two objects, the sum of all possible compositions of maps δ ~~be~~ from one to the other is 0. A morphism is a map between these guys, and its differential is the sum of all possible combinations of differential maps and morphism maps.

The cone(f) is the complex $X \xrightarrow{f} Y$.