

Derived Categories (Ref: Paul's book, notes on ~~wiki~~ wiki)

Defn: A (strictly unital) A_∞ category \mathcal{A} is a set of objects $Ob \mathcal{A}$ and a collection of graded vec spaces $hom_{\mathcal{A}}(X_0, X_1)$ for any pair $X_0, X_1 \in Ob(\mathcal{A})$, with composition maps $\forall d \geq 1$,

$$\mu_{\mathcal{A}}^d : hom(X_{d-1}, X_d) \otimes \dots \otimes hom(X_0, X_1) \rightarrow hom(X_0, X_d) [2-d]$$

Satisfy usual A_∞ equations. Also, $\exists e_x \in hom(X, X)$ ~~unit maps~~
~~multiples~~ s.t.

1) $\mu_{\mathcal{A}}^1(e_x) = 0$

2) $\pm \mu_{\mathcal{A}}^2(e_{x_1}, a) = a = \mu_{\mathcal{A}}^2(a, e_{x_0}) \quad \forall a \in hom_{\mathcal{A}}(X_0, X_1)$

3) $\mu_{\mathcal{A}}^d(a_{d-1}, \dots, e_{x_{n-1}}, \dots, a_1) = 0 \quad \forall d \geq 2, \forall n \quad a_k \in hom(X_{k-1}, X_k)$

Defn: $\mathcal{A} = A_\infty$ category, $H(\mathcal{A})$ - cohomological category associated to \mathcal{A} .

$$Ob(H(\mathcal{A})) = Ob(\mathcal{A})$$

$$H(hom(X_0, X_1), \mu_{\mathcal{A}}^d) = Hom_{H(\mathcal{A})}(X_0, X_1)$$

Assoc. for $d=1 \Rightarrow (\mu^1)^2 = 0$

Discard all homs of deg $\neq 0$. $H^0(\mathcal{A})$.

Want to define $D(\mathcal{A})$, \mathcal{A} an A_∞ category

Old

\mathcal{A} abelian cat.

\mathcal{A} A_∞ cat

$C^*(\mathcal{A}), K^*(\mathcal{A})$

$\text{Tw } \mathcal{A}$ twisted complexes

chain cx \uparrow
 \uparrow hqy class
 \uparrow morphisms

$D(\mathcal{A})$

$$H^0(\text{Tw } \mathcal{A}) = D(\mathcal{A})$$

triangulated

Defn: An A_∞ functor between A_∞ categories $F: \mathcal{A} \rightarrow \mathcal{B}$

• map on sets $\mathcal{G}: Ob \mathcal{A} \rightarrow Ob \mathcal{B}$

• $\forall d \geq 1 \quad F^d: hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \dots \otimes hom_{\mathcal{A}}(X_0, X_1) \rightarrow hom_{\mathcal{B}}(F X_0, F X_d) [1-d]$

• F^d satisfy ... relations

• $F^1(e_x) = e_{\mathcal{G}(x)}$

• $F^d(\dots, e_{x_n}, \dots) = 0 \quad \forall d \geq 2, \forall n$.

$$\mu' \mathcal{F}' = \mathcal{F}' \mu'$$

For $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$

$$H^0(\mathcal{G}): H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$$

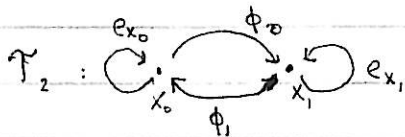
$$\text{For } [a], H^0(\mathcal{G})[a] = [\mathcal{F}'(a)]$$

Say \mathcal{F} full/faithful/quasi-equivalence if $H(\mathcal{F})$ is full/faithful/
an equivalence.

Thm: $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$ is a quasi-equivalence. Then $\exists \mathcal{G}: \mathcal{B} \rightarrow \mathcal{A}$ a

quasi-eg s.t. $\mathcal{F} \circ \mathcal{G} \cong \text{Id}_{\mathcal{B}}$

$$\mathcal{G} \circ \mathcal{F} \cong \text{Id}_{\mathcal{A}}$$



$$\mu^2(\phi_1, e_{x_1}) = \phi_1$$

$$\mu^2(e_{x_0}, \phi_1) = \phi_1$$

etc.

$$\mu^2(\phi_1, \phi_0) = e_{x_0}$$

$$\mu^2(\phi_0, \phi_1) = e_{x_1}$$

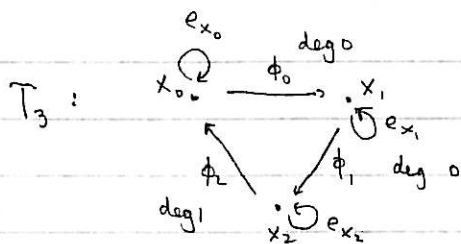
Ex: \mathcal{T}_2 is \mathcal{A}_∞ category.

Defn: $X_0, X_1 \in \mathcal{A}$ are isomorphic if $\exists \mathcal{A}_\infty$ functor $\mathcal{F}: \mathcal{T}_2 \rightarrow \mathcal{A}$

$$x_0 \mapsto X_0$$

$$x_1 \mapsto X_1$$

In $H(\mathcal{A})$, $X_0 \cong X_1$.



$$\mu^3(\phi_2, \phi_1, \phi_0) = e_0 \quad (\text{others } 0)$$

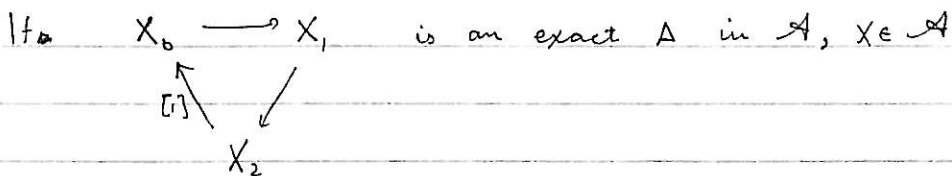
$$\mu^3(\phi_0, \phi_2, \phi_1) = e_1$$

$$\mu^3(\phi_1, \phi_0, \phi_2) = e_2$$

Defn: An exact triangle in \mathcal{A} is the image of $F: T_3 \rightarrow \mathcal{A}$.

Rmk: If $G: \mathcal{A} \rightarrow \mathcal{B}$ is an A_∞ functor, $F: T_3 \rightarrow \mathcal{A}$ is an exact Δ .
Then $G \circ F: T_3 \rightarrow \mathcal{B}$ is an exact Δ in \mathcal{B} .

Thus any A_∞ functor is exact.



Then \exists exact sequence

$$\text{Hom}_{H(\mathcal{A})}(X, X_0) \rightarrow \text{Hom}_{H(\mathcal{A})}(X, X_1) \rightarrow \text{Hom}_{H(\mathcal{A})}(X, X_2) \rightarrow \dots$$

exact.

$$\rightarrow \text{Hom}^d(X, X_0) \rightarrow \text{Hom}^d(X, X_1) \rightarrow \text{Hom}^d(X, X_2) \rightarrow \text{Hom}^{d+1}(X, X_0) \rightarrow \dots$$

and exact

$$\dots \rightarrow \text{Hom}(X_2, X) \rightarrow \text{Hom}(X_1, X) \rightarrow \text{Hom}(X_0, X) \rightarrow \dots$$

Defn: An A_∞ cat. \mathcal{A} is triangulated if it's "closed under shifts" and every morphism can be completed to an exact triangle.

Tw A

* Defn: A twisted complex in \mathcal{A} is a pair (X, δ_X)

* $X = \bigoplus_{i \in I} V^i \otimes X^i$ V^i graded vec. spaces fin dim

* $\delta_X \in \text{hom}_{\text{Tw } \mathcal{A}}^1(X, X)$ $X^i \in \mathcal{A}$
 $|I| < \infty$

$\sum_{d \geq 1} \mu^d(\delta_X \dots \delta_X) = 0$ ~~Just formal tensor of homs in V and X^i .~~

gen. Maurer-Cartan eqn.

$\text{hom}_{\text{Tw } \mathcal{A}}(X, Y) = \bigoplus_{ij} \text{Hom}(V^i, W^j) \otimes \text{hom}_{\mathcal{A}}(X^i, Y^j)$
 ~~$\otimes \text{hom}_{\mathcal{A}}(X^i, Y^j)$~~

$\mu_{\text{Tw } \mathcal{A}}^d(a_d, \dots, a_1) = \sum_{i_0 \dots i_d} \mu^{d+i_0+\dots+i_d}(\underbrace{\delta_{X_{i_d}} \dots \delta_{X_{i_1}}}_{i_d}, a_d, \delta_{X_{d-1}} \dots, \delta_{X_{d-1}}, a_{d-1}, \dots, \delta_{X_0} \dots \delta_{X_0})$

"twisting μ^d by δ^i "

Fact: $\text{Tw } \mathcal{A}$ is a triangulated \mathcal{A}_∞ category.

Defn: $D(\mathcal{A}) = H^0(\text{Tw } \mathcal{A})$

- triangulated category in the "old sense".

Shifts: $X = \bigoplus_{i \in I} V^i \otimes X^i$

$SX = X[1] = \bigoplus_{i \in I} V^i[1] \otimes X^i$

$S\delta_X = \delta_X[1]$.

Triangles \leftrightarrow define cones of morphisms

$f: X \rightarrow Y$ $\mu^1(f) = 0$ (we only care about completing triangles if $\mu^1(f) = 0$ so the cohomological ~~category~~ category is triangulated)

$\text{Cone}(f) = (X[1] \oplus Y, \begin{pmatrix} \delta_X[1] & 0 \\ -f & \delta_Y \end{pmatrix})$

~~cone~~

Fact: $X \xrightarrow{f} Y$

$$\begin{array}{ccc} & & \\ & \nearrow & \\ [1] & & \\ & \searrow & \\ & & \text{Cone}(f) \end{array}$$

is an exact triangle

$$\curvearrowright X \circlearrowleft e_x \quad \mu^2(e_x, e_x) = e_x$$

$$\text{Tw } \mathcal{A} \ni X = V \otimes X$$

\uparrow graded vec. space

δ_x - differential on V

$\text{Tw } \mathcal{A} = \text{dg vec. spaces over } k \quad e: V \rightarrow W$

$$\mu'_{\text{Tw } \mathcal{A}}(f) = \delta_x f \pm f \delta_x$$

~~Wk~~ Homomorphisms in $H(\text{Tw } \mathcal{A})$ will be ~~to~~ homotopy equivalence classes of chain maps

$$\Rightarrow H^0(\text{Tw } \mathcal{A}) = K(\text{Vect}_k) = D(\text{Vect}_k)$$

\uparrow homotopy category

* Way to think about this: a twisted complex is a collection of objects with differentials & a sequence of maps between them "going to the right" somehow. ~~The sum~~ For every two objects, the sum of all possible compositions of maps δ ~~be~~ from one to the other is 0. A morphism is a map between these guys, and its differential is the sum of all possible combinations of differential maps and morphism maps.

The cone(f) is the complex $X \xrightarrow{f} Y$.