Hochschild Cohomology

Outline: I. HH for ordinary (assoc.) algebra
- relation to deformation theory
- intrinsic formality

II. A_∞ (algebra, category) generalisation

III. Map $QH^*(M) \to HH^*(\mathcal{F}(M), \mathcal{F}(M))$

Review:

A = assoc, A_∞ algebra, tensor co-algebra $TA = \sum A^{\otimes i}$ comult Δ

$Q : TA \to A \underset{\pi_A}{\overset{\text{extend}}{\rightleftharpoons}} \hat{Q} : TA \to TA$

require $\hat{Q}$ to be a coderivation w.r.t. Δ (uniquely specifies $\hat{Q}$).

Namely:

$$\hat{Q} := \sum 1^{\otimes i} \otimes Q^k \otimes 1^{\otimes j}$$

(up to sign)

An A_∞ alg. struct. on A is a map $Q = \sum m_i : TA[1] \to A[1]$

$Q^2 = 0$ $\quad A_\infty$ equations are $Q \circ \hat{Q} = 0$

If A assoc., $Q = 0 + p + 0 + \dots \quad \hat{Q}^2 = 0 \Leftrightarrow A$ assoc.

Restrict to (A, p) an assoc. algebra.

Hochschild chain complex is

$CC^r(A, M)$ A-module

$= \mathrm{Hom}(A^{\otimes r}, M)$

$\delta : CC^r(A, M) \to CC^{r+1}(A, M)$

$$\delta\varphi^r(a_1, \dots, a_{r+1}) = \sum (-1)^* \varphi^r(a_1, \dots, p(a_i, a_{i+1}), \dots, a_{r+1}) + p(a_1, \varphi^r(a_2, \dots, a_{r+1})) - p(\varphi^r(a_1, \dots, a_r), a_{r+1})$$

Check: $\delta^2 = 0$, $\quad HH^r(A, M) \quad HH^r(A, A)$

There's a bracket on CC^*

$[\,,\,] : CC^r \times CC^s \to CC^{r+s-1}$

$$\varphi^r, \eta^s \longmapsto \sum (-1)^* \varphi^r(\_\_\_\ \eta^s(\_\_\ )\ \_\_\ \_\_)$$

$$+ \sum (-1)^{**} \eta^s(\_\ \ \varphi^r(\_\_)\ \_\_\ \_\_)$$

Observe: If $Q = 0 + p + 0 + \dots : TA \to A$ then $\delta = [\cdot, Q]$. In particular, $[\,,\,]$ descends to HH^*. Gives CC^* the structure of a dg Lie (super) algebra.

E.g. "$[\varphi^r, \phi^s] = -(-1)^{(r-1)(s-1)} [\phi^s, \varphi^r]$"

Note $[Q, Q] = 0$ (associativity).

<u>Lemma</u>: If p_t is a deformation of p

e.g. $p_t : A[[t]] \times A[[t]] \to A[[t]]$

$p_t = p_0 + \sum p_i t^i$

p_t associative $\Leftrightarrow \frac{1}{2}[p_t, p_t] = 0$

<u>Pf</u>: Trivial from equation for $[\,,\,]$

$\frac{1}{2}[p_t, p_t] = 0$ collect powers of t

t^0: $[p_0, p_0] = 0$ ✓

t^1: $[p_0, p_1] = 0$ $\quad \delta p_1 = 0 \Rightarrow p_1$ a Hochschild cocycle

t^2: $[p_0, p_2] + \frac{1}{2}[p_1, p_1] = 0$

i.e. $\delta p_2 + \frac{1}{2}[p_1, p_1] = 0$

...

t^1 tells us 1st order deformations are given by HH^2 (in fact they are in 1-1 correspondence - Hochschild coboundaries give isomorphic things)

t^2 tells us we need to be able to find p_2 s.t.

$$\delta p_2 = \tfrac{1}{2}[p_1, p_1]$$

(can always do this if $HH^3 = 0$).

Intrinsic formality

Defn: An assoc. algebra A is intrinsically formal if for any A_∞ algebra B with $H(B) \cong A$ as algebras, B is formal.

Thm (Kadeishvili): $HH^*(A, A[2-q]) = 0$ for $q > 2$ $\Rightarrow A$ is intrinsically formal.

Pf: (sketch) Take A_∞ B-alg, $H(B) \cong A$ as algebras.

By ~~Hom~~ Perturbation Lemma, get an A_∞ alg. structure on A with $m_1 = 0$, $m_2 = p$ s.t. ϕ is a quasi-iso.

Inductively assume $m_i = 0$ for $2 < i < k$

Then the $(k+1)$-input A_∞ equation for A reads

$$\sum m_k(\_\_\ m_2(\_\_)\ \_\ \_) + m_2(m_k(\_), \_) + m_2(\_, m_k(\_)) = 0$$

i.e. m_k is a Hochschild cocycle in $HH^k(A, A[2-k])$

In particular, $m_k = \delta\eta$

Use this to construct a quasi-iso killing m_k

(ex: idea: $F: TA \to A$ with $F_1 = id$ (formal diffeomorphisms)

$F_1 = id$

$F_2 \ldots F_{k-2} = 0$

$F_{k-1} = \eta$

$F_k, \ldots = \text{anything}$

$\Rightarrow$ resulting A_∞ structure has $m_k = 0$).

* Paul: tells you you can stop work...?

A_∞ algebra generalisation

Let (A, m_i) be an A_∞ algebra. The total Hochschild complex

$$CC^*(A) := \mathrm{Hom}(TA, A)$$
$$= \bigoplus_r \hom(TA, A[r])$$

As before, there's a ~~product~~ bracket $[\,,\,]$ on $CC^*(A)$ (Gerstenhaber bracket), in the exact same way.

On $CC^*(A)$ we can see this as

$$[\varphi, \eta] := \pi_A(\hat{\varphi} \circ \hat{\eta} - \hat{\eta} \circ \hat{\varphi})$$

graded commutator

$$Q = m_1 + m_2 + \dots : TA \longrightarrow A$$

Differential (Hochschild):

$$\delta\varphi := [\varphi, Q].$$

Now, HH tells us the same information about A_∞ deformations.

i.e. Q_t a deformation is $A_\infty \iff \frac{1}{2}[Q_t, Q_t] = 0$

Maurer–Cartan framework

Rmk about gradings

$$CC^r(A) = \bigoplus_d \hom(A^{\otimes d}, A[r-d])$$
$$Q_t = Q_0 + Q_1 t + \dots : TA \longrightarrow A.$$

A_∞ categories

Let $\mathcal{A}$ be an A_∞ category. An element of $CC^r(\mathcal{A}, \mathcal{A})$ is a set of data $\{h^i\}_0^\infty$ where h^d is, for each $(d+1)$-tuple of objects $(L_0, \dots, L_d)$

$$h^d_{(L_0, \dots, L_d)} : \mathrm{Hom}(L_{d-1}, L_d) \otimes \dots \otimes \mathrm{Hom}(L_0, L_1) \longrightarrow \mathrm{Hom}(L_0, L_d)[r-d]$$

Hochschild differential

"bracket with ~~xx~~ $\sum m_i$'s"

$$\delta\{h^i\} = \{\hat{h}^i\}$$

where $\hat{h}^i_{(L_0 \dots L_d)} = \sum h^j_{(L_0 \dots L_k\, L_s \dots L_i)}(\dots m^{s-k}_{(L_k \dots L_s)} \dots)$ + opposite term

i inputs

(m on outside)

Ex: $HH^*(\mathcal{A}, \mathcal{A}) = H(\hom_Q(id, id))$

Q is the A_∞ category of endofunctions of $\mathcal{A}$.

A map from $QH^*(M)$ to $HH^*(\mathcal{F}(M), \mathcal{F}(M))$.

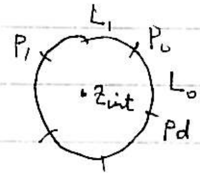
(FO^3 chap. 3) $\quad q_1 : QH^*(M) \longrightarrow HH$

$$\cancel{b}\ b \longmapsto \cancel{\text{consider}}\ \{\varphi^i\}$$

$$\varphi^d : Hom(L_{d-1}, L_d) \otimes \dots \otimes Hom(L_0, L_1) \longrightarrow Hom(L_0, L_d)$$

$$p_1 \otimes \dots \otimes p_d \longmapsto \sum n^b(p_0, \dots, p_d)\, p_0$$

where $\displaystyle n^b(p_0, \dots, p_d) = \sum_{\substack{\cancel{\beta \in \pi_2(M, L)} \\ (\beta \in \pi_2(M, L_0, \dots, L_d))}} t^{\omega(\beta)} \# \mathcal{M}^b_\beta (L_0, \dots, L_d, p_0, \dots, p_d)$

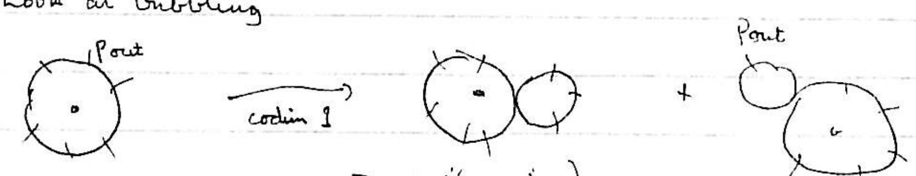
Where $\mathcal{M}^b_\beta = \Big\{ u :$  $\xrightarrow[J\text{-hol}]{} M \qquad z_{int} \longmapsto B \Big\}$

Pick a cycle B representing $PD(b)$
$n-r$ dimensional.

If $b \in QH^r$ then this is a Hochschild cocycle of degree r.

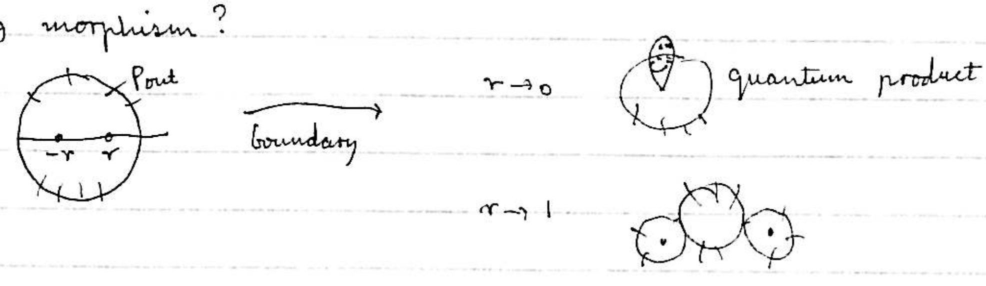
Why a Hochschild cocycle?

Look at bubbling



$$\sum \varphi^i(\_\ m^j\ \_) + m^j(\_\ \varphi^i\ \_\ \_\ \_) = 0$$

Ring morphism?



HH product is $\varphi \circ \psi = \sum m_i (\_\_ \varphi^j (\_) - - - \varphi^k (- - -) - - -)$

This is an iso sometimes...